

# Optimal Contracting with Unobservable Managerial Hedging\*

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# Optimal Contracting with Unobservable Managerial Hedging

## Abstract

We develop a continuous-time model where a risk neutral principal contracts with a CARA manager who is protected by limited liability in that managerial compensation can never be negative. The project output can be increased by costly unobservable managerial effort, but the project can also be liquidated if the manager decides to quit due to limited liability. In addition, the manager can trade a market portfolio and a risk-free bond in an unobservable private account. We study the interactions among performance evaluation, unobservable costly managerial effort, and unobservable managerial saving and hedging. New to the literature, our unique model setup permits compatibility of private saving and hedging, manager's risk aversion, and inefficient project liquidation in one contracting problem. The inefficient project liquidation endogenously induces effective risk aversion of the risk-neutral principal. Consequently, the principal does not completely filter out market return from the manager's contract. In the optimal contract, the sensitivity to market return balances the principal's risk sharing behavior and incentive provision motive. Compared with a contract of only relative or absolute performance evaluation, our optimal contract delivers higher value for the principal. In a nutshell, our model provides a new dynamic mechanism to explain the mixed empirical evidence on relative performance evaluation. The sensitivity to market return in our model is positive near the liquidation boundary, where negative market shock leads the project closer to its liquidation. This is consistent with the empirical evidence of the heightened managerial turnover in bad market conditions. Finally, we implement the optimal contract by risk management accounts, private debt and private equity in an entrepreneurship context.

**Keywords:** Moral hazard, managerial hedging, optimal contracting, performance evaluation, capital structure

**JEL Codes:** G11, G12, G32, D82, E2

# 1 Introduction

Equity-based incentives constitute a considerable part of managerial compensations.<sup>1</sup> One important theoretical prediction from Holmstrom (1982) is that market shocks should be removed from managerial compensations because an individual manager is not able to affect market return and it is costly for managers to shoulder market risk. Therefore, it is efficient for investors to compensate managers based on relative performance evaluation (RPE), which benchmarks firm output against market return. Holmstrom (1982) assumes that it is prohibitively expensive for managers to hedge in their private accounts.<sup>2</sup> However, empirical literature documents that managers do hedge against excessive risk exposures in their compensation packages by trading stocks, indices, or using derivative contracts.<sup>3</sup> The hedging behavior, which thereafter we refer to as *managerial hedging*, can partially offset, or even eliminate, the incentives embedded in compensation packages, hence presents a challenge for the investors to offer appropriate incentives.<sup>4</sup>

To study the effects of managerial hedging in private account on optimal contracting, we develop a continuous-time model where a risk-neutral principal (investor) contracts with a risk-averse agent (manager) to operate a project.<sup>5</sup> The agent can increase expected output of the project by exerting costly hidden effort. Meanwhile, the agent can perform unobservable saving and hedging by trading a market portfolio and a risk-free bond in her private account without any friction. The uncertainty of the project output can be decomposed into marketwide shocks and idiosyncratic shocks, where the former can be hedged by trading the market portfolio. Importantly, the agent is protected by limited liability: she cannot subsidize project cash flows by receiving negative compensations from the principal. To maximize project value, the principal designs a compensation contract based on both the project output performance and the market return. We disentangle the interactions among performance evaluation, unobservable costly managerial effort, and unobservable managerial

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<sup>1</sup>Guay (1999), for instance, estimates median stock-based CEO wealth amounts to 6.79 million dollars for 1000 largest companies in Compustat in 1993. Burns, Jindra, and Minnick (2017) report that equity-based compensation ratio is 20% for private firms that succeed in IPO in Capital IQ data set.

<sup>2</sup>More recently, Cvitanic, Henderson, and Lazrak (2014) consider optimal contracting with observable managerial hedging and conclude that RPE contract is optimal. Gao (2010) highlights that the managerial hedging can profoundly influence corporate governance and capital structure.

<sup>3</sup>Ofek and Yermack (2000) report that firm managers respond to an increase of equity compensation by selling previously held shares for the purpose of diversification. Bettis, Bizjak, and Lemmon (2001) find that corporate managers trade zero-cost collars and equity swaps to hedge against excessive ownership.

<sup>4</sup>Jin (2002) studies the relationship between the pay-performance sensitivity and systematic and idiosyncratic risks. Garvey and Milbourn (2003) show that cost of managerial hedging and cost of benchmarking are important factors in RPE contracts.

<sup>5</sup>Thereafter we use agent and manager interchangeably.

saving and hedging.

The limited liability restriction has important economic implications. Since the contract payment is constrained to be non-negative and the agent can invest in the capital market and consume from her private account even without the contract, the agent is furnished with a voluntary retirement option when the contract value drops to zero. If the agent decides to quit, the project will be liquidated. Similar to DeMarzo and Sannikov (2006), the contractual termination in the form of project liquidation leads to endogenous risk aversion for the risk-neutral principal (i.e., concavity of the principal's value function). Endogenous risk aversion generates risk sharing motive for the principal to optimally share market risk with the agent. On the other hand, the principal has incentive to reduce the degree of moral hazard by filtering out market return from the project output so as to elicit managerial effort. The trade-off between risk sharing and incentive provision results in an optimal dynamic contract which features a mixture of absolute performance evaluation (APE) and relative performance evaluation (RPE) whereby the principal does not completely filter out all market risk from the managerial contract. Therefore, our model provides a new dynamic mechanism to explain the limited empirical support of RPE in the literature.<sup>6</sup>

Our work is related to several other works which have been proposed to explain the lack of RPE in practice. Aggarwal and Samwick (1999a) suggest that strategic interaction between firms can produce positive contract sensitivity to market. Oyer (2004) proposes an explanation of APE based on manager's outside career opportunities. Ou-Yang (2005) shows that market component is not completely filtered out from managerial compensations due to risk sharing motive of the risk-averse investors, and that positive contract sensitivity to market return appears if the principals are sufficiently risk-averse. Ozdenoren and Yuan (2017) allow performance evaluation components to respond to economic cycle and obtain an optimal mixture of APE and RPE. However, both Ou-Yang (2005) and Ozdenoren and Yuan (2017) assume risk-averse principals and obtain static contract sensitivities. Our model shows that optimal mixture of REP and APE arises in a dynamic setting even with risk neutral principal.

One important feature of our optimal contract is that the sensitivity to market return is positive

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<sup>6</sup>See, e.g., Antle and Smith (1986), Barro and Barro (1990), Jensen and Murphy (1990), Janakiraman, Lambert, and Larcker (1992), Aggrwal and Samwick (1999b), Jenter and Kanaan (2015), and so on. In particular, Janakiraman, Lambert, and Larcker (1992) find evidence on a positive relationship between managerial compensation and industry performance based on accounting data. Aggarwal and Samwick (1999b) find little evidence that the market shocks are completely removed from executive compensations, but record that market pay-performance sensitivity can be positive. Nevertheless, the last decade witnesses increasingly prevalent practice of benchmarking managerial compensation against peers, which is regarded as evidence of RPE; see, e.g., Gong, Li, and Shin (2011), Albuquerque, De Franco, and Verdi (2013), and Bizjak et al. (2017).

near the project liquidation. The implication is that a negative market shock can lead the project closer to its liquidation and cause the manager to retire, which explains the negative relation between CEO turnover and industry or market performance found in Jenter and Kanaan (2015).

One consequence of our optimal contract is that the agent invests in the market portfolio to hedge uncertainty in her compensation and restores exactly the mean-variance efficient portfolio in total certainty equivalent, which is similar to static models like Jin (2002) and Garvey and Milbourn (2003). However, managerial hedging in the agent's private wealth induces exactly opposite effects on her contract value, and impacts the probability of project liquidation. Therefore, we find that unobservable managerial hedging is detrimental to the principal's value even in the optimal contract.

After we have thoroughly analyzed the optimal contract and examined its implications, following the literature such as DeMarzo and Sannikov (2006) and Biais et al. (2007), we proceed to implement the optimal contract by a capital structure which consists of risk management accounts, private debt and private equity in the context of entrepreneurship. Particularly, the risk management accounts accomplish the principal's risk sharing motive. Dynamic balance sheet, values and market prices of securities are derived and analyzed.

Methodologically, we make a couple of contributions. First, we employ CARA utility of the agent to surmount the difficulty of unobservable savings in dynamic contracting. It is eminently noted in the literature that optimal contracting with private savings should be treated with special vigilance owing to the agent's *double-deviating* strategy. Since private hedging in our model embeds private savings, we resort to CARA utility of the agent to bypass the stumbling block of private savings under general preference and focus on the effects of managerial hedging, under performance evaluations. The reason is that, with CARA utility, the agent's marginal utility under optimal strategies is always proportional to her indirect utility.<sup>7</sup> Thus, the agent's utility process (equivalently, the agent's contract value process in our model) can still serve as the unique state variable for the principal in optimal contracting with private savings. The tractability of CARA utility also allows us to separate wealth effect from incentive provision. As a result, our optimal contract is independent of the agent's private wealth, but only depends on the agent's contract value. This implies that agents with heterogeneous wealth will pool in the same contract, which is an interesting implication to mechanism design.

Second, we contribute to dynamic contracting literature by incorporating agent's risk aversion, private saving and hedging, and inefficient liquidation in one model. The existing literature has

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<sup>7</sup>The advantage of CARA utility to address the difficulty caused by private savings has been availed by He (2011) and Williams (2015).

addressed any combinations of two of the three features. To see clearly, Figure 1 highlights the most relevant papers that have incorporated two of the three features. He (2011), Williams (2015), and Di Tella and Sannikov (2016) model risk-averse agent and her private saving behavior. Their models do not feature inefficient liquidation.<sup>8</sup> Inefficient liquidation and private savings are compatible in DeMarzo and Sannikov (2006) and He (2009) with risk neutral agents. Without private savings, Sannikov (2008) derives optimal contract with both risk-averse agent and inefficient liquidation. To our best knowledge, our model that combines all three features is new to literature.

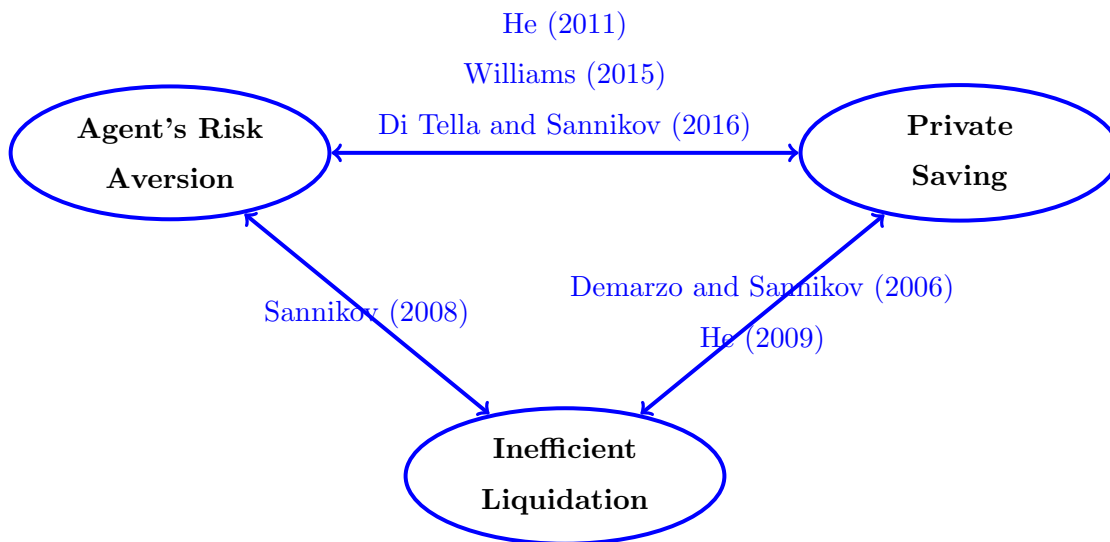


Figure 1: Pairwise combinations of modeling features

The rest of the paper is organized as follows. Section 2 formally states the market environment and our model setup and then presents the methodology to solve the model. Section 3 examines implications of the model solution and compares our optimal contract with pure APE or RPE contract. In Section 4, we implement the optimal contract with familiar securities in an entrepreneurship context. Both inside values and market prices of securities are derived there.<sup>9</sup> The paper concludes in Section 5. Technical conditions, proofs, and numerical details are provided in the Appendix as Section 6.

<sup>8</sup>Di Tella and Sannikov (2016) model unobservable managerial saving and hedging under CRRA utility. Although the CRRA agent is protected by limited liability, it never binds since the project size can be scaled down by the principal. So they do not derive project liquidation.

<sup>9</sup>Inside values refer to values derived from discounting future cash flows by the venture capital's discount rate while market prices refer to present values derived from using the market martingale pricing operator. See Section 4.2 and Section 4.3 for details.

## 2 The Model

### 2.1 Model Environment

To initiate and operate the project, the principal offers a take-it-or-leave-it contract to the agent. This project generates an output or cash flows with the following dynamics

$$dX_t = (\mu + A_t)dt + \rho\psi dB_t + \sqrt{1 - \rho^2}\psi dB_t^\perp, \quad (1)$$

where  $B$  represents marketwide shocks, and  $B^\perp$  stands for idiosyncratic risk of the project. They are assumed to be independent standard Brownian motions. The constant  $\mu$  represents the instantaneous output of the project without the agent's input, and the constant  $\psi$  describes the project volatility. The agent is able to increase the project output by exerting costly effort  $A$ , which takes values in a finite interval  $\mathcal{A}$ . The cost of effort, described in monetary unit, is given by

$$h(A) = \frac{\kappa}{2}A^2 + bA \quad (2)$$

with  $\kappa > 0$  and  $b > 0$ .<sup>10</sup> The principal knows  $\mathcal{A}$  and observes  $X$  continuously in time. However, the principal cannot identify the agent's effort. This unobservable managerial effort introduces the first source of moral hazard in our model.

There is a capital market in the economy. The market consists of a risk-free bond with continuous rate  $r$  and a market portfolio whose return follows

$$dR_t = mdt + \sigma dB_t. \quad (3)$$

The constant  $\rho$  in output (1) represents instantaneous correlation between the project output and the market return. The market return is observed continuously by both the agent and the principal. The agent can also participate in the capital market to trade the bond and the market portfolio continuously without friction. Even though the principal can filter out the market component in output (1) through his observation of the market return, it is impossible for him to separate the agent's effort from the idiosyncratic risk. Hence,  $\sqrt{1 - \rho^2}\psi$  measures the severity of information asymmetry. On the flip side, when  $\rho \neq 0$ , the agent can trade the market portfolio to hedge against uncertainty in her compensation contract and potentially offset the incentives that the contract provides. Therefore,  $\rho$  also measures the easiness of hedging for the agent.

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<sup>10</sup>In dynamic contracting literature, monetary cost is commonly used together with CARA utility for the agent; see, e.g., Holmstrom and Milgrom (1987) and He (2011). The quadratic form of cost function is used in Figure 1 of Sannikov (2008) and satisfies conditions of Proposition 6 in Strulovici and Szydlowski (2015).

To compensate the agent for her effort, the principal pays her a salary whose cumulative value is described by a non-decreasing process  $I_t$ . We assume that  $dI_t \geq 0$  so that the agent in our model is protected by *limited liability*: the agent cannot subsidize the principal by accepting negative compensation. This limited liability assumption is a pivotal driver of our results and differentiates our model from those in He (2011) and Williams (2015).

Starting from an initial wealth  $\hat{s}$ , if the agent invests  $\pi$  (in monetary unit) in the market portfolio and put the remaining wealth in the risk-free bond, while receiving payments from the principal, deducting effort cost, and consuming at a rate of  $c$ , then the agent's private wealth process follows

$$dS_t = rS_t dt + \pi_t(m - r)dt + \pi_t \sigma dB_t + dI_t - h(A_t)dt - c_t dt, \quad S_0 = \hat{s}. \quad (4)$$

Note that we do not restrict the agent's private wealth to be positive. A savings account in deficit means that the agent borrows money to support her consumption and portfolio choice. The agent's private wealth process, including the initial wealth  $\hat{s}$ , is assumed to be *unobservable* by the principal and thus not contractible. Thereby, the agent's saving and hedging actions are hidden from the principal. This unobservable managerial saving and hedging is the second source of moral hazard in our model. The coexistence of unobservable effort and private savings allows the agent to perform *double-deviating strategy* of shirking and saving simultaneously. He (2012) prevents the double-deviating strategy by keeping the agent's marginal utility non-increasing. Di Tella and Sannikov (2016) show that the key to alleviate a CRRA agent's private savings incentive is to lower the agent's risk exposure after bad performance. To elude the stumbling block caused by private savings, He (2011) and Williams (2013, 2015) separate the income effect and the incentive provision by working with CARA agents. However, these papers, except He (2012) and Di Tella and Sannikov (2016), do not impose limited liability restrictions on contract payments, while in He (2012) and Di Tella and Sannikov (2016), the limited liability restrictions are not binding in the optimal contracts. Without limited liability, or when it is not binding, the agent can delegate the principal to save and invest for her, so that the principal's contract choice set can be reduced to one including only no-saving and no-investment contacts; see, e.g., Lemma 2 in He (2011) and Lemma 19 in Di Tella and Sannikov (2016). After this reduction, (an equivalent form of) the optimal contract can be obtained under which the agent's optimal strategy is not to save nor invest. However, when the limited liability restriction is binding, this reduction of contract space cannot be performed in our model, because the negative increments of the agent's wealth process cannot be rebated by the principal via compensation with non-negative increments. In order to tackle the agent's unobservable wealth and focus on the effects of managerial hedging, we assume that the agent's preference is described by a CARA utility with the risk aversion parameter  $\gamma$ , i.e.,



$u(c) = -\frac{1}{\gamma}e^{-\gamma c}$ . This allows the principal to separate the agent's private wealth from the incentive provision.

Due to the agent's limited liability, our model also features a contractual termination in the form of project liquidation. If the agent does not work for the principal after retirement, we assume that she still has access to the capital market and consumes from her private account. Then the agent's problem is a standard Merton's optimal consumption and investment problem

$$\sup_{c, \pi} \mathbb{E} \left[ \bar{\delta} \int_0^\infty e^{-\bar{\delta}t} u(c_t) dt \right],$$

subject to

$$dS_t = rS_t dt + \pi_t(m - r)dt + \pi_t \sigma dB_t - c_t dt. \quad (5)$$

A standard dynamic programming argument yields that the agent's value function is  $u(rS - \ell)$ , which is the agent's reservation utility in that the contract cannot lead to lower utility for the agent and wherein

$$\ell = \frac{1}{\gamma} \left( 1 - \frac{\bar{\delta}}{r} - \ln \left( \frac{r}{\bar{\delta}} \right) \right) - \frac{(m-r)^2}{2r\gamma\sigma^2}.^{11}$$

Therefore, the agent voluntarily retires from the project at a stopping time  $\tau$  when her continuation value from working is no more than her reservation value. It follows then that the agent's optimization problem is

$$\sup_{A, c, \pi} \mathbb{E} \left[ \bar{\delta} \int_0^\tau e^{-\bar{\delta}t} u(c_t) dt + e^{-\bar{\delta}\tau} u(rS_\tau - \ell) \right],^{12} \quad (6)$$

with her participation constraint

$$\mathcal{G}_t > rS_t - \ell, \quad \text{for all } t \leq \tau. \quad (\text{PC})$$

Here  $\mathcal{G}$  is the agent's certainty equivalent process, i.e.

$$u(\mathcal{G}_t) = \text{esssup}_{A, c, \pi} \mathbb{E}_t \left[ \bar{\delta} \int_t^\infty e^{-\bar{\delta}(s-t)} u(c_s) ds \right], \quad (7)$$

where  $\mathbb{E}_t[\cdot]$  denotes the conditional expectation,  $\text{esssup}$  stands for the essential supremum, and

$$\tau = \inf \{ t \geq 0 : \mathcal{G}_t \leq rS_t - \ell \}.$$

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<sup>11</sup>The certainty equivalent wealth  $rS - \ell$  for the Merton's problem can be easily retrieved from equations (61), (62), and (63) in Merton (1969).

<sup>12</sup>The control problem in (4) and (6) is equivalent to  $dS_t = rS_t dt + \pi_t(m - r)dt + \pi_t \sigma dB_t + dI_t - \tilde{c}_t dt$  and  $\sup_{A, \tilde{c}, \pi} \mathbb{E} \left[ \bar{\delta} \int_0^\tau e^{-\bar{\delta}t} u(\tilde{c}_t - h(A_t)) dt + e^{-\bar{\delta}\tau} u(rS_\tau - \ell) \right]$  by setting  $c = \tilde{c} - h(A)$ . That is,  $c$  is the consumption  $\tilde{c}$  net of the cost of effort  $h(A)$  to yield the agent's utility  $u(c)$ .

When the agent retires at time  $\tau$ , the project is liquidated. Then the agent achieves her reservation value  $u(rS - \ell)$  and the principal is left with a project liquidation value  $\Phi$ .

The principal is assumed to be risk neutral. Observing the project output, his problem is to maximize the expected value of cash flows less the compensation payment to the agent,

$$\sup_I \mathbb{E} \left[ \delta \int_0^\tau e^{-\delta t} [(\mu + A_t)dt - dI_t] + \delta e^{-\delta \tau} \Phi \right], \quad (8)$$

subject to the agent's participation constraint (PC) and her incentive compatibility constraint

$$A \text{ in (8) is the optimal effort for (6).} \quad (\text{IC})$$

We require a technical regularity condition that

$$r > \delta.$$

Economically, it ensures that compensation to the agent will not be deferred forever. A similar condition is assumed in DeMarzo and Sannikov (2006).

## 2.2 The Agent's Problem

Given payment process  $I_t$ , we solve the agent's optimization problem in this subsection. Treating the compensation process as a random endowment, the agent is confronted with an optimal consumption and investment problem in an incomplete market. Define the agent's continuation utility as

$$\Upsilon_t = \text{esssup}_{A,c,\pi} \mathbb{E}_t \left[ \bar{\delta} \int_t^\tau e^{-\bar{\delta}(s-t)} u(c_s) ds + e^{-\bar{\delta}(\tau-t)} u(rS_\tau - \ell) \right], \quad (9)$$

where  $S_t$  follows (4). The CARA form of  $u$  permits us to define a process  $Y_t$  via

$$\Upsilon_t = -\frac{1}{\gamma} e^{-\gamma(rS_t - \ell + rY_t)}, \quad (10)$$

where the exponent on the right-hand side is the agent's certainty equivalent

$$\mathcal{G}_t = rS_t - \ell + rY_t. \quad (11)$$

Therefore, the process  $Y_t$ , which is the certainty equivalent provided by the contract (normalized by  $r$ ), captures the net rent enjoyed by the agent from the contract. Accordingly, we term  $Y_t$  as the agent's *contract value*. In terms of the contract value  $Y_t$ , the stopping time of project liquidation can be restated as

$$\tau = \inf\{t \geq 0 : Y_t \leq 0\}. \quad (12)$$

**Remark 2.1** *Note that we have assumed that the agent has no outside option. If she has an outside option with a certainty equivalent  $\underline{y}$ , the stopping time of project liquidation will be given by*

$$\tau = \inf\{t \geq 0 : Y_t \leq \underline{y}\}. \quad (13)$$

### 2.3 The Contract Space

In what follows, we will derive heuristically the dynamics of  $Y$ . To this end, recall that the principal can observe and contract on both the output  $X$  and the market return  $R$ . We first consider the absolute performance evaluation (APE) where the compensation depends on total project output. The contract value is specified as

$$dY_t = dH_t + Z_t dX_t, \quad (\text{APE})$$

where  $dX_t$  is the total output in (1) and  $H$  is a finite-variation process to be determined later. The process  $Z_t$ , determined by the principal as a part of the incentive contract, specifies the sensitivity of the agent's contract value with respect to the absolute project output.

Another contract form is to *completely* remove the market influence in the output. We define the relative output as

$$d\widehat{X}_t = dX_t - \frac{\rho\psi}{\sigma} dR_t.$$

The relative performance evaluation (RPE) is then specified as

$$dY_t = dH_t + Z_t d\widehat{X}_t = dH_t + Z_t dX_t - \frac{\rho\psi}{\sigma} Z_t dR_t, \quad (\text{RPE})$$

where  $H$  is another finite-variation process to be determined and  $Z_t$  is the sensitivity of the agent's contract value with respect to the relative project output.<sup>13</sup>

Generally, the principal may choose to filter part (not zero as in APE and not completely as in RPE) of the market influence in the output. Let

$$d\widetilde{X}_t = dX_t - a_t dR_t,$$

where  $a_t$  is a choice of the principal. Using  $d\widetilde{X}_t$ , The performance evaluation is specified as

$$dY_t = dH_t + Z_t d\widetilde{X}_t = dH_t + Z_t dX_t - a_t Z_t dR_t, \quad (\text{PE})$$

where  $H$  is a finite-variation process to be determined later.

Obviously, (PE) contains both (APE) and (RPE) as special cases by setting  $a_t = 0$  and  $a_t = \rho\psi/\sigma$ , respectively. Setting  $U_t = -a_t Z_t$ , we can then rewrite (PE) as

$$dY_t = dH_t + Z_t dX_t + U_t dR_t = dH_t + Z_t d\widehat{X}_t + \zeta_t dR_t, \quad (14)$$

where

$$\zeta = \frac{\rho\psi}{\sigma} Z + U \quad (15)$$

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<sup>13</sup>Note that to filter out  $\rho\psi dB_t$  from the output in (1), we should define  $d\widehat{X}_t$  as  $d\widehat{X}_t = dX_t - \frac{\rho\psi}{\sigma} (dR_t - mdt)$ . But the term  $\frac{\rho\psi}{\sigma} mdt$  can be absorbed into  $dH_t$ .

describes the contract gross exposure to marketwide shocks. So the principal exposes the agent's contract value to both  $X$  and  $R$  by choosing sensitivities  $Z$  and  $U$  respectively. We designate the contract form in (14) as the *optimal performance evaluation* (OPE). Clearly, it is a mixture of APE and RPE. The processes  $Z$  and  $U$  are contract sensitivities with respect to  $X$  and  $R$ , respectively.

## 2.4 The Agent's Optimal Choices

Given the contract sensitivities  $Z$  and  $U$  in (14) which will be determined in the principal's problem in the next section, the agent chooses the optimal strategy  $(A, c, \pi)$ .

From the dynamic programming principle we know that  $e^{-\bar{\delta}t}\Upsilon_t + \bar{\delta} \int_0^t e^{-\bar{\delta}s}u(c_s)ds$  is a supermartingale for an arbitrary strategy  $(A, c, \pi)$ , and is a martingale until time  $\tau$  for the agent's optimal strategy. Using (4) and (14), we obtain from Itô's formula that the drift of  $e^{-\bar{\delta}t}\Upsilon_t + \bar{\delta} \int_0^t e^{-\bar{\delta}s}u(c_s)ds$  (divided throughout by  $-r\gamma e^{-\bar{\delta}t}\Upsilon_t$ ) is

$$\begin{aligned} & dH_t + dI_t + \left\{ \frac{\bar{\delta}}{r\gamma} + rS_t + \pi_t(m-r) - h(A_t) - c_t + Z_t(\mu + A_t) + U_t m \right. \\ & \left. - \frac{\gamma r}{2} \left[ \pi_t^2 \sigma^2 + (\rho\psi Z_t + \sigma U_t)^2 + 2\pi_t \sigma (\rho\psi Z_t + \sigma U_t) + (1 - \rho^2)\psi^2 Z_t^2 \right] - \frac{\bar{\delta}u(c_t)}{r\gamma\Upsilon_t} \right\} dt. \end{aligned}$$

Maximizing over  $A$ ,  $c$ , and  $\pi$ , the previous drift should be zero, leading to

$$\begin{aligned} dH_t = & -dI_t + \left[ -\frac{\bar{\delta}}{r\gamma} - rS_t - Z_t\mu - U_t m + \frac{\gamma r}{2} \left( (\rho\psi Z_t + \sigma U_t)^2 + (1 - \rho^2)\psi^2 Z_t^2 \right) \right] dt \\ & - \sup_{A, c, \pi} \left\{ \pi_t(m-r) - h(A_t) - c_t + Z_t A_t - \frac{\gamma r}{2} \left[ \pi_t^2 \sigma^2 + 2\pi_t \sigma (\rho\psi Z_t + \sigma U_t) \right] \right. \\ & \left. - \frac{\bar{\delta}u(c_t)}{r\gamma\Upsilon_t} \right\} dt. \end{aligned} \tag{16}$$

From the first-order conditions, we obtain the agent's optimal portfolio position

$$\pi^* = \frac{m-r}{r\gamma\sigma^2} - \frac{1}{\sigma}(\rho\psi Z + \sigma U), \tag{17}$$

and optimal consumption rate

$$c^* = rS - \ell + rY - \frac{1}{\gamma} \ln\left(\frac{r}{\bar{\delta}}\right). \tag{18}$$

Define  $h^*(Z) = \min_{A \in \mathcal{A}} \{h(A) - ZA\}$  as the convex conjugate of  $h$  and denote

$$A^*(Z) = \arg \min h^*(Z), \tag{19}$$

which is the agent's optimal effort choice. Plugging (17), (18), and (19) into (16), we obtain

$$dH_t = \left[ rY_t - rU_t - Z_t \left( \mu - \frac{(m-r)\rho\psi}{\sigma} \right) + \frac{r\gamma}{2} \psi^2 (1 - \rho^2) Z_t^2 + h^*(Z_t) \right] dt - dI_t. \tag{20}$$

Combining (14) and (20), the agent's contract value  $Y$  has the following dynamics

$$dY_t = \left[ rY_t + \frac{r\gamma}{2}\psi^2(1-\rho^2)Z_t^2 + h(A_t^*) + (m-r)\zeta_t \right] dt - dI_t + \zeta_t \sigma dB_t + Z_t \sqrt{1-\rho^2} \psi dB_t^\perp. \quad (21)$$

Evidently, the agent's contract value compensates her for additional consumption  $rY$ , effort cost  $h(A^*)$ , and idiosyncratic risk exposure  $\psi^2(1-\rho^2)Z^2$ , but introduces an extra market risk exposure  $\zeta dR$ . Note from (17) that  $-\zeta$  is the agent's hedging demand. This means that the agent's optimal portfolio  $\pi^*$  first offsets the exposure to the market risk in her contract value (14), and then restores Metron's (1969) optimal portfolio  $\frac{m-r}{r\gamma\sigma^2}$ . Briefly speaking, the agent's hedging demand is caused by the additional market risk exposure  $\zeta dR$  to her contract value  $Y$ .

The following result verifies the optimality of  $\pi^*$ ,  $c^*$ , and  $A^*$  in (17), (18), and (19). Before that, let us first define the admissible class of strategies. Given the payment process  $I$ , a strategy  $(A, c, \pi)$  is *admissible* to the agent if there exist  $Z$  and  $U$ , adapted to the filtration generated by  $(X, R)$ , such that

$$\Upsilon_t^{A,c,\pi,Z,U} = -\frac{1}{\gamma} e^{-\gamma(rS_t - \ell + rY_t)}, \quad (22)$$

where  $S$  satisfies (4) and  $Y$  follows (21), satisfies the transversality condition

$$\lim_{\tilde{\tau} \rightarrow \infty} \mathbb{E} \left[ e^{-\delta\tilde{\tau}} \Upsilon_{\tilde{\tau}}^{A,c,\pi,Z,U} 1_{\{\tilde{\tau} \leq \tau\}} \right] = 0, \quad \text{for any stopping time } \tilde{\tau}, \quad (23)$$

where  $\tau = \inf\{t \geq 0 : Y_t \leq 0\}$ . The proof of the following lemma is deferred to Section 6.5 in the Appendix.

**Lemma 2.2** *Suppose that the strategy  $(A^*, c^*, \pi^*)$  is admissible. Then it is the optimal strategy for the agent among all admissible strategies, i.e.,*

$$\begin{aligned} \mathbb{E} \left[ \bar{\delta} \int_0^\tau e^{-\bar{\delta}t} u(c_t^*) dt + e^{-\bar{\delta}\tau} u(rS_\tau - \ell) \right] &= \sup_{A,c,\pi} \mathbb{E} \left[ \bar{\delta} \int_0^\tau e^{-\bar{\delta}t} u(c_t) dt + e^{-\bar{\delta}\tau} u(rS_\tau - \ell) \right] \\ &= -\frac{1}{\gamma} e^{-\gamma(r\hat{s} - \ell + rY_0)}. \end{aligned}$$

The admissibility of  $(A^*, c^*, \pi^*)$  will be verified when the principal's problem is analyzed in Section 6.6 in the Appendix.

## 2.5 The Principal's Problem

After the agent's problem is solved, we take the agent's contract value  $Y_t$  in (21) as the state variable for the principal's optimization problem to select both the optimal contract sensitivity  $Z^*$  and the optimal gross market exposure  $\zeta^*$ . Comparing to DeMarzo and Sannikov (2006), Sannikov (2008), He (2011), and Williams (2013) wherein the agent's continuation utility serves as the state variable,

we take the agent's certainty equivalent less her private wealth as the state variable, because the agent's private wealth is unobservable.

The principal's optimization problem is

$$W(\hat{y}) = \sup_{I, Z, \zeta} \mathbb{E} \left[ \delta \int_0^\tau e^{-\delta t} [(\mu + A^*(Z_t))dt - dI_t] + \delta e^{-\delta \tau} \Phi \right], \quad (24)$$

where  $A^*(Z)$  is given by (19),  $Y$  follows (21) with initial condition  $Y_0 = \hat{y} \geq 0$ .

We first follow Miao and Rivera (2016) to show that the compensation  $dI_t$  is an impulse function. To this end, let  $dI_t = i_t dt$  with  $i_t \geq 0$  for all  $t \geq 0$ . The principal's value function  $W$  satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} \delta W(y) = & \sup_{i \geq 0, z \in \mathcal{Z}, \zeta \in \Gamma} \left\{ \delta(\mu + A^*(z) - i) + \left[ ry + \frac{r\gamma}{2} \psi^2(1 - \rho^2)z^2 + h(A^*(z)) + (m - r)\zeta - i \right] W'(y) \right. \\ & \left. + \frac{1}{2} \left[ \sigma^2 \zeta^2 + (1 - \rho^2) \psi^2 z^2 \right] W''(y) \right\}, \end{aligned} \quad (25)$$

where  $\mathcal{Z}$  and  $\Gamma$  are compact sets on real line, but  $i$  is allowed to take unbounded value. The term with  $i$  on the right hand side is given by

$$f(i) = -(\delta + W'(y))i.$$

To have a finite solution for  $W(y)$ , it is obvious that we must have  $W'(y) \geq -\delta$ ; otherwise,  $f(i)$  is unbounded for any unbounded  $i$  and thus  $W(y)$  becomes unbounded too. For  $W'(y) > -\delta$ ,  $f(i)$  achieves the highest value (zero) with  $i = 0$ . Thereby,  $i$  can be nonzero (i.e.,  $i > 0$ ) only when  $W'(y) = -\delta$ . Define

$$\bar{y} = \inf\{y \geq 0 : W'(y) = -\delta\}.$$

So the principal's optimal payment strategy is to postpone any payment until the agent's contract value reaches an upper bound  $\bar{y}$  and make a minimum payment at  $\bar{y}$  to reflect process  $Y$  back to the left of  $\bar{y}$ . In consequence, the optimal payment process is an impulse type,<sup>14</sup> and  $\bar{y}$  specifies a payment boundary condition

$$W'(\bar{y}) = -\delta. \quad (26)$$

Since the boundary point  $\bar{y}$  is a free boundary, similar to DeMarzo and Sannikov (2006), the super-contact condition

$$W''(\bar{y}) = 0 \quad (27)$$

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<sup>14</sup>In a model without private saving and hedging where consumption only comes from compensation, impulse payment cannot possibly be optimal because the risk-averse agent prefers smooth consumption. In our model with private saving and hedging, smooth consumption depends on the sum of the agent's private wealth and her contract value, regardless of the payment or compensation form. The payment form is then optimally determined from the principal's maximization problem. The proof here shows that the optimal payment takes the form of impulse type.

is required to identify  $\bar{y}$ . On the other hand, when the project is liquidated at time  $\tau$ , the principal recovers the project liquidation value  $\Phi$ . This leads to the liquidation boundary condition,<sup>15</sup>

$$W(0) = \delta\Phi. \quad (28)$$

To sum up, when  $y \in (0, \bar{y})$ , we have  $dI_t = 0$ , a binding limited liability restriction in our model. In this *continuation region*, the HJB equation (25) for the principal's problem is simplified to

$$\begin{aligned} \delta W(y) = & \sup_{z \in \mathcal{Z}, \zeta \in \Gamma} \left\{ \delta(\mu + A^*(z)) + \left[ ry + \frac{r\gamma}{2}\psi^2(1 - \rho^2)z^2 + h(A^*(z)) + (m - r)\zeta \right] W'(y) \right. \\ & \left. + \frac{1}{2} \left[ \sigma^2\zeta^2 + (1 - \rho^2)\psi^2z^2 \right] W''(y) \right\}. \end{aligned} \quad (29)$$

The following proposition presents the optimal contract, and its proof is deferred to Section 6.6 in the Appendix.

**Proposition 2.3** *Under the optimal contract, the agent's contract value  $Y$  evolves according to (21), with an initial contract value  $\hat{y} \in [0, \bar{y}]$ . The optimal contract sensitivities are determined via  $Z = Z^*(Y)$  and  $U = U^*(Y) = \zeta^*(Y) - \frac{\rho\psi}{\sigma}Z^*(Y)$ , where  $Z^* \in \mathcal{Z}$  and  $\zeta^* \in \Gamma$  are the maximizers of*

$$\delta A^*(Z) + \left[ \frac{r\gamma}{2}\psi^2(1 - \rho^2)Z^2 + h(A^*(Z)) + (m - r)\zeta \right] W'(Y) + \frac{1}{2} \left[ \sigma^2\zeta^2 + (1 - \rho^2)\psi^2Z^2 \right] W''(Y).$$

*If  $\hat{y} > \bar{y}$ , the principal pays an immediate lump-sum compensation  $\hat{y} - \bar{y}$  to the agent. When  $Y \in [0, \bar{y})$ , no compensation is paid. At  $Y = \bar{y}$ , compensation is paid to reflect  $Y$  to the left of  $\bar{y}$ . The project is liquidated when  $Y$  drops to 0, i.e.,  $Y_\tau = 0$ . The principal's value is determined by the HJB equation (29) when  $y \in (0, \bar{y})$  and  $W'(y) = -\delta$  when  $y \geq \bar{y}$ , together with boundary conditions (26), (27), and (28).*

We call the contract specified by  $(Y, Z^*, U^*, I)$  in Proposition 2.3 the *optimal performance evaluation* (OPE) contract. The results in Proposition 2.3 show that with the agent's limited liability restriction and private savings, the optimal contract takes the form of impulse payment and specifies an inefficient liquidation boundary. Thereby, our model incorporates agent's risk aversion, private saving and hedging, and a liquidation boundary in a unified framework, which is, to our best knowledge, new in dynamic contracting literature.

Proposition 2.3 also provides an interesting implication to contractual mechanism design. Since the agent's saving and investment are unobservable and not contractible, the optimal contract in

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<sup>15</sup>Note that the principal's valuation function in (24) is scaled by  $\delta$ . If the agent has an outside option as in Remark 2.1, the corresponding condition becomes  $W(\underline{y}) = \delta\Phi$ .

Proposition 2.3 does not depend on the agent’s private wealth and investment.<sup>16</sup> For any two agents who have the same preference and effort cost but different private wealth, if the principal offers a contract  $(Y, Z^*, U^*, I)$ , then both agents will accept the contract. In other words, different agents with heterogenous wealth will pool in the same OPE contract.

**Remark 2.4** *Before we leave this section, we comment that in a model without private saving and hedging where consumption only comes from compensation, an impulse type payment can be optimal only for a risk-neutral agent since a risk-averse agent prefers smooth consumption. In our model with private saving and hedging, consumption in (18) is smooth and depends on the sum of private wealth  $S$  and the contract value  $Y$ , regardless of the payment form  $dI$ , which is optimally determined from the principal’s maximization problem. Our proof shows that the optimal  $dI$  for the risk-averse agent is still an impulse type in our model.*

### 3 The Optimal Contract

#### 3.1 Risk Sharing, Incentive Provision, and OPE

In order to understand the optimal contract, we start with the HJB equation (29). First, similar to DeMarzo and Sannikov (2006) and Sannikov (2008), we are able to show that the  $W''(y) < 0$  in the continuation region  $(0, \bar{y})$ .<sup>17</sup> So the principal is implicitly risk-averse. Another important observation from (29) is that  $Z$  and  $\zeta$  are chosen separately by the principal. According to (15), for an arbitrary choice of  $Z \in \mathcal{Z}$ , the principal can balance the impact of  $Z$  in  $\zeta$  by selecting some  $U$  to achieve the desired value of  $\zeta$ . Thus, the choice of  $\zeta$  does not influence the agent’s effort and its associated cost. Figure 1 plots numerical solutions to the model under three different values of  $\rho$ . It is shown in Panel A that as the degree of information asymmetry, measured by  $\sqrt{1 - \rho^2}\psi$ , decreases, the the principal’s value uniformly increases,<sup>18</sup> and the continuation region shrinks. Intuitively, a project with less information asymmetry requires less financial liquidity and thus the principal prepares less room between liquidation and compensation. A similar feature is found in DeMarzo and Sannikov (2006).

Figure 1 about here

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<sup>16</sup>The irrelevance of the agent’s reservation value is caused by CARA utility, reminiscent of Theorem 2 in static model of Holmstrom and Milgrom (1987). What matters is the certainty equivalent provided by the contract.

<sup>17</sup>This is proved in Proposition 6.4 in the Appendix.

<sup>18</sup>In addition, define  $y^* = \arg \max_{y \geq 0} W(y)$ . By Feynman-Kac formula, we can show that  $\partial_\rho W(y^*; \rho) > 0$ .



The optimal choice of  $\zeta$  in (29) is given by

$$\zeta^* = -\frac{m-r}{\sigma^2} \frac{W'(y)}{W''(y)}, \quad (30)$$

if not binding. Obviously,  $\zeta^*$  resembles Merton's (1969) portfolio position of a CARA investor and is the gross market exposure of the contract. The term  $-\frac{W''(y)}{W'(y)}$  depicts the principal's risk attitude. The endogenous risk aversion of the principal motivates him to share market risk with the agent. If  $W'(y) > 0$ , the principal takes a positive gross market exposure and the agent is rewarded by the market return in her contract value  $Y$  in (21). If  $W'(y) \leq 0$ , the principal chooses a non-positive gross market exposure which benchmarks the agent against market return in her contract value. In consequence, risk sharing generates benefit

$$(m-r)\zeta^*W'(y) + \frac{1}{2}\sigma^2\zeta^{*2}W''(y) = \frac{1}{2}(m-r)\zeta^*W'(y)$$

for the principal. In Panel B of Figure 1, we plot the principal's optimal gross market exposure.

To choose optimal contract sensitivity  $Z^*$ , the principal maximizes

$$\delta A^*(Z) + \left[ \frac{r\gamma}{2}\psi^2(1-\rho^2)Z^2 + h(A^*(Z)) \right] W'(y) + \frac{1}{2}(1-\rho^2)\psi^2Z^2W''(y). \quad (31)$$

Clearly, a direct benefit of the APE component  $Z^*dX$  is to elicit the agent's effort input  $\delta A^*(Z)$ . To compensate the agent for effort, the principal has to reimburse her for effort cost  $h(A^*)$  which is evaluated by the principal's marginal value at  $-h(A^*)W'(y)$ . In addition, since the agent is risk-averse, the principal needs to compensate her for idiosyncratic risk exposure whose cost is  $\frac{r\gamma}{2}\psi^2(1-\rho^2)Z^2$  evaluated by the agent and  $-\frac{r\gamma}{2}\psi^2(1-\rho^2)Z^2W'(y)$  measured by the principal's marginal value. Finally, because the principal is implicitly risk-averse, he also suffers from value loss caused by idiosyncratic risk exposure, which is equal to  $-\frac{1}{2}\frac{W''(y)}{W'(y)}\psi^2(1-\rho^2)Z^2W'(y)$ . In a nutshell, sensitivity  $Z^*$  plays the role of incentive provision. Panel C in Figure 1 shows that  $Z^*$  is bounded from below near the liquidation boundary in order to maintain a zero effort level. When the agent's contract value is low, imposing low sensitivity to output effectively reduces project liquidation probability by introducing low idiosyncratic risk exposure in  $Y$ . When the agent's contract value increases,  $Z^*$  surges until a summit and finally decreases. Since  $W''(y)$  increases but  $W'(y)$  decreases and turns from positive to negative in our numerical results, the non-monotonic sensitivity can be explained by opposite functional directions of  $W'(y)$  and  $W''(y)$ : the risk exposure cost to the principal (the third term in (31)) dominates when the agent's contract value is relatively low, while the incentive provision cost (the second term in (31)) dominates when the agent's contract value is high.

Once  $Z^*$  and  $\zeta^*$  are determined, the benchmarking component in the OPE contract is determined by

$$U^* = \zeta^* - \frac{\rho\psi}{\sigma} Z^*. \quad (32)$$

Economically, the optimal sensitivity to market return is equal to the principal's optimal gross market exposure less marketwide shocks introduced from the output. Thus it reflects the principal's trade-off between his risk sharing behavior and incentive provision motive. This trade-off leads to an optimal mixture of APE and RPE. Panel D of Figure 1 plots dynamics of  $U^*$  as a result of the trade-off.

### 3.2 The Agent's Optimal Effort and Wealth Dynamics

We now turn to incentive compatible strategies of the agent. According to (19), the agent's optimal effort  $A^*(Z)$  is determined by the contract sensitivity to the project output. In particular,  $A^*(Z) = (h')^{-1}(Z)$  in the interior of  $\mathcal{A}$ . Therefore, the agent's optimal effort displays the same pattern as the optimal contract sensitivity  $Z^*$ , which is plotted in Panel A of Figure 2. Intuitively, higher sensitivity incentivizes higher effort but also increases incentive cost and the probability of inefficient liquidation. So, it can be observed that optimal effort is highly dynamic and non-monotonic in our model even under CARA utility, which is in stark contrast to most principal-agent models under CARA preference such as Holmstrom and Milgrom (1987), Ou-Yang (2003), and Williams (2015). Besides, if information asymmetry is more grievous (i.e.,  $\rho$  is smaller), the principal would like to promote lower effort.

Figure 2 about here

If the agent does not enter into the contract or if she retires, her certainty equivalent (normalized by  $r$ ) is  $S - \frac{\ell}{r}$ , which evolves according to

$$dS_t = \left[ \frac{(m-r)^2}{2r\gamma\sigma^2} + \frac{1}{\gamma} \left( 1 - \frac{\bar{\delta}}{r} \right) \right] dt + \frac{m-r}{r\gamma\sigma} dB_t. \quad (33)$$

When the agent works for the principal under the optimal contract and if her contract value is inside the continuation region, her certainty equivalent (normalized by  $r$ ) is  $S - \frac{\ell}{r} + Y$ , which follows

$$d(S_t + Y_t) = \left[ \frac{(m-r)^2}{2r\gamma\sigma^2} + \frac{1}{\gamma} \left( 1 - \frac{\bar{\delta}}{r} \right) \right] dt + \frac{m-r}{r\gamma\sigma} dB_t + \frac{r\gamma}{2} \psi^2 (1 - \rho^2) (Z_t^*)^2 dt + Z_t^* \sqrt{1 - \rho^2} \psi dB_t^\perp. \quad (34)$$

Comparing the above two processes, we draw the conclusion that the genuine incentive is provided through idiosyncratic risk exposure by  $\frac{r\gamma}{2} \psi^2 (1 - \rho^2) (Z_t^*)^2 dt + Z_t^* \psi \sqrt{1 - \rho^2} dB_t^\perp$ , where  $\sqrt{1 - \rho^2} \psi$  measures the degree of agency friction. Note that the agent's certainty equivalent has a continuous

path. This is because the impulse compensation  $dI$  in the dynamics of  $S$  in (4) cancels out the term  $dI$  in the dynamics of  $Y$  in (21).

The agent's optimal consumption is presented in (18). Clearly, its path is smooth even though the compensation  $dI$  is impulse type. The constant term  $\ell - \frac{1}{\gamma} \ln\left(\frac{r}{\delta}\right)$  adjusts the consumption rate according to the difference between risk-free rate and the agent's discount rate. The remaining part of the agent's optimal consumption is her certainty equivalent. With fixed private wealth  $S$ , Panel B of Figure 2 plots the agent's optimal consumption rate under the optimal contract and after retirement. The optimal consumption rate under the optimal contract grows at the risk-free rate with the agent's contract value, and it is constant without the contract or after retirement. Given the agent's optimal consumption  $c^*$  in (18), the agent's marginal utility is  $u'(c^*) = -\frac{\gamma r}{\delta} \Upsilon$ . Hence, the agent's marginal utility follows

$$d\Upsilon_t = (\bar{\delta} - r)\Upsilon_t dt - \frac{m-r}{\sigma} \Upsilon_t dB_t - r\gamma\psi\sqrt{1-\rho^2}Z_t^* \Upsilon_t dB_t^\perp, \quad (35)$$

wherein the term  $r\gamma\psi\sqrt{1-\rho^2}Z_t^*$  represents the agent's idiosyncratic risk premium. Particularly, the agent's marginal utility is a martingale when  $\bar{\delta} = r$ .

Equation (17) shows that agent's optimal portfolio position is exactly the Merton's (1969) result  $\frac{m-r}{r\gamma\sigma^2}$  less gross market exposure  $\zeta^*$  in her contract value. And the agent's hedging demand  $-\zeta^*$  is intended to offset additional exposure introduced by her contract value. Comparing the dynamics of the agent's total wealth with and without the optimal contract in (33) and (34), we can see that the purpose of the agent's managerial hedging behavior is to target volatility of her certainty equivalent loaded on marketwide shocks  $dB$  at  $\frac{m-r}{r\gamma\sigma}$ , which is reminiscent of hedging under CRRA utility like in Ingersoll (2006), Ang, Papanikolaou, and Westerfield (2014). Panel C of Figure 2 plots the agent's optimal positions invested in the market portfolio under three values of  $\rho$ , together with the case after her retirement. Under the optimal contract, given  $\rho$ , since  $\zeta^*$  is positive if  $W'(y) > 0$  but turns to be negative when  $W'(y) < 0$ , the manager voluntarily decreases her risk exposure to marketwide shocks when her contract value is small but leverages up when her contract value grows up.

### 3.3 Model Comparison

In order to understand the optimal contract, we now hold a horse race among APE, RPE, and OPE contracts.<sup>19</sup> The flexibility of controlling  $\zeta$  and  $Z$  independently is not possible when the principal

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<sup>19</sup>A model with APE and one with RPE are solved and briefly analyzed in Section 6.2 and 6.3 respectively in the Appendix.

uses only APE or RPE. So pure APE and RPE contracts are restricted solutions to the principal’s problem, and they will deliver less value to the principal than the OPE contract, which is confirmed in Panel A of Figure 3. Panel A also shows that the RPE contract improves the principal’s value comparing to the APE contract, because the RPE contract reduces the information asymmetry by filtering out the market return.

Figure 3 about here

Panel B of Figure 3 compares the RPE component across models. In the APE contract,  $U \equiv 0$  by definition. In the RPE contract,  $U = -\frac{\rho\psi}{\sigma}Z$ . In either case, the RPE component is non-positive. In our optimal contract,  $U^* = \zeta^* - \frac{\rho\psi}{\sigma}Z^*$ , which takes into account the principal’s risk sharing incentive and causes the OPE contract to be superior to the RPE contract. In effect, the principal does not use the RPE component to completely remove market risk from the contract value. When the agent’s contract value is close to the liquidation boundary,  $U^*$  can even be positive: a negative shock to the market return decreases the agent’s contract value and increases project liquidation probability. Therefore, our model predicts that the contract could be terminated if market performs badly, which is consistent with the empirical evidence found in Jenter and Kanaan (2015). In a general equilibrium model with risk-averse principal, Ou-Yang (2005) observes that contract sensitivity to market return can be positive, when the principal is sufficiently risk averse. In our model, the principal is risk neutral. However, inefficient project liquidation and endogenous risk aversion of the principal engender the positive sensitivity  $U^*$  as well. Moreover, in contrast to the static contract sensitivity, our contract sensitivity is state-dependent and changes sign with the agent’s contract value, which can partially explain why the empirical analyses cannot unambiguously identify the prevalence of RPE contracts. Our model mechanism concerning OPE is also different from that in Ozdenoren and Yuan (2017) who inquire into the effects of contractual externalities and allow their APE component to be either positive or negative so as to rectify the agent’s idiosyncratic risk exposure.

In Panel C of Figure 3, we plot the agent’s effort choice under three models. It is shown that the RPE contract motivates higher effort than the APE contract due to lower agency friction. On the left half of the state space, the agent’s effort under OPE is higher than that in a model with only APE or RPE. However, as the contract value increases, the marginal cost of incentive provision rises, making the principal incentivize the agent to exert lower effort under the OPE contract.

Finally, we consider a model where the agent’s investment position is contractible.<sup>20</sup> In general, it costs the principal  $-\frac{r\gamma}{2}\sigma^2 \left[ \pi - \left( \frac{m-r}{r\gamma\sigma^2} - \zeta \right) \right]^2 W'(y)$  to impose a contractible investment strategy

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<sup>20</sup>The model with contractible managerial investment is delineated in Section 6.4 in the Appendix.

$\pi$ . Recall from (17) that  $\pi^* = \frac{m-r}{r\gamma\sigma^2} - \zeta$ . When  $W'(y) \leq 0$ , the previous cost is minimized at  $\pi = \pi^*$ . Therefore the agent's hedging demand aligns her interest with the principal's value. When  $W'(y) > 0$ , however, it is better off for the principal to impose an investment position  $\pi$  as far as possible from  $\pi^*$ . In this case, the unobservable managerial hedging is detrimental to the principal's value. Thereby, when the agent's investment position is contractible, the principal can improve his value by dictating any  $\pi \neq \pi^*$  if  $W'(y) > 0$  and  $\pi = \pi^*$  otherwise. Figure 3 corroborates our analyses by presenting the case where  $\pi = 0$  if  $W'(y) > 0$  and  $\pi = \pi^*$  if  $W'(y) \leq 0$ .

## 4 Capital Structure Implementation

### 4.1 Security Design

Now, we situate our model in a realistic environment. Because both the agent and the principal evaluate their claims in the contract under the augmented probability measure generated by  $(X, R)$ , they are confronted with an incomplete market. Hence, we project our model on the context of entrepreneurship where the principal represents a venture capital (or private equity) while the agent acts as a project manager (or entrepreneur). Particularly, the manager is protected by limited liability and the risk neutral venture capital discounts cash flows at rate  $\delta < r$ . Assume that the project requires an initial investment  $K$  to start. In order to initiate the project, at date 0, the manager resorts to the venture capital to raise money and set up the entrepreneurial venture. Accordingly, financial securities are issued to the venture capital in support of the project and to compensate the manager.<sup>21</sup> The optimal contract derived in Proposition 2.3 is said to be implemented by these securities if the cash flows generated by the project can completely reimburse claims of these securities on condition that both the manager and the venture capital voluntarily participate into the project and that the manager's incentive compatibility constraint is satisfied. Previous works like DeMarzo and Sannikov (2006) and Biais et al. (2007) have demonstrated that the implementation of such contract is not unique in similar environments, since the cash flows can be artificially sliced up and assigned to different securities as long as the entire value of all securities sums up to the entrepreneurial venture's value, a revisit to capital structure irrelevance principle (Modigliani and Miller (1958)). With little loss of generality, we mainly refer to Biais

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<sup>21</sup>It is irrelevant whether the manager devotes part of her private wealth into the project as skin in the game before date 0. If she contributes  $\chi$  from her initial wealth  $\hat{s}$  into the project, we can replace  $\hat{s}$  by  $\hat{s} - \chi$  and  $K$  by  $K - \chi$ . So  $K$  is net financing raised.

et al. (2007) to implement the OPE contract by risk management accounts,<sup>22</sup> bonds, and stocks. In an entrepreneurship environment, these securities are not publicly traded in the market and thus their values under venture capital investment are different from market prices. We term these values *inside values* which are evaluated by the partners in the venture capital. Importantly, one proviso to the capital structure implementation is that  $0 < Z^* \leq 1$  in the continuation region.<sup>23</sup> We present our implementation in the following proposition and defer its proof until Section 6.8 in the Appendix.

**Proposition 4.1** *Consider the following capital structure to finance the project. At date 0, the entrepreneurial venture issues bonds with face value  $F$  and stocks to a venture capital at inside values in order to support  $K$  and construct two risk management accounts with initial deposit  $Y_0$  and  $\frac{1-Z_0^*}{Z_0^*}Y_0$  respectively. When  $t > 0$ , the bonds distribute floating coupon payment*

$$x_t = \mu + A_t^* - \frac{h(A_t^*)}{Z_t^*} - \frac{r\gamma}{2}\psi^2(1 - \rho^2)Z_t^* - \frac{m-r}{\sigma}\rho\psi \quad (36)$$

*continuously to the debt partners in the venture capital if  $Y_t \in (0, \bar{y}]$ .<sup>24</sup> At  $Y_t = \bar{y}$ , stocks distribute dividend*

$$dL_t = \frac{dI_t}{Z^*(\bar{y})}, \quad (37)$$

*among which the manager owns  $Z^*(\bar{y})dL_t$  and the equity partners in the venture capital entitles  $(1 - Z^*(\bar{y}))dL_t$ . In the first (resp. second) risk management account, money amount  $U_t^*$  (resp.  $(1 - Z_t^*)U_t^*/Z_t^*$ ) is invested in the market portfolio and the remaining  $Y_t - U_t^*$  (resp.  $(1 - Z_t^*)(Y_t - U_t^*)/Z_t^*$ ) is invested in the risk-free bond. When  $Y_t \in (0, \bar{y}]$ ,  $Z_t^*$  fraction of operation losses (or profits) is counted into account  $Y_t$  while the remaining  $1 - Z_t^*$  fraction is accrued on account  $\frac{1 - Z_t^*}{Z_t^*}Y_t$ . Finally, the project is liquidated on condition that either account runs out of money. In case of liquidation, the venture capital confiscates the project whose liquidation value is  $\Phi$  for the venture capital. Then, this capital structure implements the optimal contract in Proposition 2.3 wherein the balance on the first account  $Y$  characterizes the agent's (manager's) contract value.*

<sup>22</sup>Note that in the OPE contract the principal has to trade the market portfolio due to his risk sharing incentive. So cash reserves in Biais et al. (2007) correspond to risk management accounts in our model. Bolton, Chen, and Wang (2011, 2013), for instance, model risk management under corporate finance context.

<sup>23</sup>The condition that  $Z^* \in (0, 1]$  can be ensured by assuming that the upper bound  $\bar{Z}$  of the admissible control set  $\mathcal{Z}$  is not larger than 1. This requirement is satisfied by the parameter values in Table 1.

<sup>24</sup>According to Kaplan and Strömberg (2003), dynamic coupon payment, as well as floating managerial ownership discussed below, is achieved by intensively used convertibles among venture capital investment. Meanwhile, venture capitals usually have considerable discretion in contract design over the distribution of cash flows in their entrepreneurial ventures. Of course, for economically meaningful interpretation of coupon payment, we have to check that  $x \geq 0$  in the continuation region.

Similar to Biais et al. (2007), under our capital structure implementation, the asset side of the entrepreneurial venture's balance sheet includes the value of the project and two risk management accounts. The liability side, on the other hand, consists of private debt and private equity. At the outset, both debt and equity are held only by the venture capital. After the establishment of the entrepreneurial venture, the incentive contract works as if the manager were promised by the venture capital with a floating managerial ownership of  $Z^*$  as inside equity which confers the privilege of dividends on the manager but absents him from any residual equity value (if any) in case of liquidation; see also Demarzo and Sannikov (2006) and He (2009) for further discussion on inside equity. Due to incentive compatibility, the dividend payout policy will be obeyed by the manager at her discretion. When the entrepreneurial venture becomes financially distressed, inside stake of the manager  $Z^*$  will be reduced in our model, which is consistent with realistic venture capital investment contract described in Kaplan and Strömberg (2003) which favours the venture capital more claims and voting rights when the entrepreneurial venture performs unsatisfactorily. On the flip side, managerial ownership is also diminished if the project operates well, because high power incentive is too costly for the venture capital then.

The two risk management accounts distinguishes our implementation from previous works.<sup>25</sup> In particular, market positions  $U^*$  and  $\frac{1-Z^*}{Z^*}U^*$  carry out the venture capital's risk sharing objective. By construction, the two risk management accounts will drain simultaneously. Therefore, liquidation will be triggered once the liquidity dries up in the entrepreneurial venture. In case of liquidation, the project is seized by the venture capital who would be the only owner and left with the liquidation value of the project  $\Phi$ , which is evaluated from the venture capital's perspective. To facilitate specifying boundary conditions to bonds and stocks, we assume that bonds are senior to stocks in that the inside value of bonds in case of liquidation is  $\min\{\Phi, F\}$  while that of stocks is  $\max\{0, \Phi - F\}$ . Since the project could be liquidated with positive probability, debt partners faces default risk.

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<sup>25</sup>We cannot combine two risk management accounts and refer only to total balance  $\frac{Y}{Z^*(Y)}$  to implement the OPE contract, since we cannot ensure that the mapping from  $\frac{Y}{Z^*(Y)}$  to  $Y$  is one-to-one due to dynamic incentive. Note that the principal needs to observe the process of  $Y$  which generalizes the entire history of the agent's performance to determine optimal sensitivities  $Z^*$  and  $U^*$ . Therefore we have to keep account of an independent balance  $Y$ .

## 4.2 Inside Values of Securities

Given the capital structure implementation, we now derive asset pricing implications of the securities in our model. Define the inside values of stocks  $P$  and bonds  $D$  respectively as

$$P_t = \mathbb{E}_t \left[ \int_t^\tau e^{-\delta(s-t)} dL_s + e^{-\delta(\tau-t)} \max \{0, \Phi - F\} \right], \quad (38)$$

and

$$D_t = \mathbb{E}_t \left[ \int_t^\tau e^{-\delta(s-t)} x_s ds + e^{-\delta(\tau-t)} \min \{\Phi, F\} \right]. \quad (39)$$

With these inside values, we represent the dynamic balance sheet (evaluated by the partners) in the entrepreneurial venture in the following proposition.

**Proposition 4.2** *At any date  $t \geq 0$ , the balance sheet of the entrepreneurial venture satisfies*

$$\frac{W(Y_t)}{\delta} + \frac{Y_t}{Z_t^*} + Q_t = D_t + P_t - Z^*(\bar{y}) \left\{ P_t - \mathbb{E}_t \left[ e^{-\delta(\tau-t)} \max \{0, \Phi - F\} \right] \right\}, \quad (40)$$

where  $Q_t = \mathbb{E}_t \left[ \int_t^\tau e^{-\delta(s-t)} \left[ (r - \delta) \frac{Y_s}{Z_s^*} + (m - r) \frac{U_s^*}{Z_s^*} \right] ds \right]$ . In particular, at the commencement of the project, we have

$$\left( \frac{W(Y_0)}{\delta} - K \right) + \left( K + \frac{Y_0}{Z_0^*} + Q_0 \right) = D_0 + P_0 - Z^*(\bar{y}) \left\{ P_0 - \mathbb{E} \left[ e^{-\delta\tau} \max \{0, \Phi - F\} \right] \right\}. \quad (41)$$

The left-hand side of (40) represents the sum of assets' values in the venture capital that can create cash flows. These assets' values incorporate value of the project and value of the risk management accounts. However, since the principal discounts cash flows at a lower rate than the risk-free interest rate, and because there are market positions in the risk management accounts, the present value of total risk management account differs from its book value  $\frac{Y}{Z^*}$ . Note that  $Q$  can be rewritten as

$$Q_t = \mathbb{E}_t \left[ \int_t^\tau e^{-\delta(s-t)} \left[ (r - \delta) \frac{Y_s - U_s^*}{Z_s^*} + (m - \delta) \frac{U_s^*}{Z_s^*} \right] ds \right].$$

Hence, variable  $Q$  adjusts this incongruity. The right-hand side of (40) balances the assets' value of the venture capital by the present value of bonds and stocks held by the venture capital. However, since the manager's inside equity endows her with a fraction  $Z^*(\bar{y})$  of total dividend but exempts her from residual liquidation value of the project, the value of the manager's inside equity, evaluated at the payout boundary, is subtracted from inside value of stocks. At date 0, issuance of securities generates a total amount of money  $\frac{W(Y_0)}{\delta} + \frac{Y_0}{Z_0^*}$  whose inside value equals  $\frac{W(Y_0)}{\delta} + \frac{Y_0}{Z_0^*} + Q_0$  for the venture capital, among which  $K$  is contributed to initial investment while  $\frac{Y_0}{Z_0^*}$  builds up two risk management accounts. The remaining  $\frac{W(Y_0)}{\delta} - K$  amounts to net surplus maintained by the



venture capital if the project is initiated. Consequently, a necessary condition for the project to be financed is given by  $\sup \{W(y) : y \in [0, \bar{y}]\} \geq \delta K$ .

Generally, there are two factors that impact values (as well as prices) of securities in our implementation. First, because the dynamics of the manager's contract value are endogenous results of the venture capital's risk sharing behavior and incentive provision motive, the dynamics of the risk management account should significantly affect the values of securities. Second, cash flows of these securities directly contribute to their values. Recall the definition of coupon and dividend in (36) and (37) respectively. It is clear that a change of  $Z^*$  will change the payment structure of the securities and thus affect their values.

Panel A of Figure 4 plots inside value of stocks  $P$  under three different values of  $\rho$ . In all three cases, the inside value of stocks increases as the first risk management account grows up. In case of liquidation, equity partners are left with nothing. Interestingly, information asymmetry does not monotonically influence inside value of stocks due to its complicated effect on the dynamics of the manager's contract value. Near the liquidation boundary, inside value of stocks decreases with  $\sqrt{1 - \rho^2}$ . However, near the payment boundary, inside value of stocks hinges more on the magnitude of dividend. And since optimal sensitivity  $Z^*$  declines with  $\sqrt{1 - \rho^2}$  there (see Panel C of Figure 1), lower  $Z^*$  leads to higher total dividend and larger discounted value in case of lower  $\rho$ .

Figure 4 about here

Panel B of Figure 4 exhibits inside value of bonds  $D$  in three cases of  $\rho$ . Clearly, inside value of bonds uniformly decreases with  $\rho$ , which reflects the effect of  $\rho$  on coupon payment (36) due to high market risk price  $\frac{m-r}{\sigma}$ . Besides, inside value of bonds in each case displays non-monotonic patterns, which again reflects complicated dynamics of coupon and is a ramification of dynamic incentive in our model. With constant managerial ownership, Biais et al. (2007), however, find that coupon payment decreases with the amount of cash reserve in their model. Note that the inside value of bonds is not necessarily monotonic near the liquidation boundary where default probability rises: it drops first from its liquidation value and then increases as the account balance expands. Intuitively, consider a sample path wherein an unfortunate manager encounters a series of negative shocks in  $\rho\psi B + \sqrt{1 - \rho^2}\psi B^\perp$  so that balance of the risk management accounts in the entrepreneurial venture abates from right to left. Due to dynamics of optimal sensitivities in Figure 1, we know that the principal spontaneously reduces risk exposure of the manager's contract value in attempt to prevent project liquidation. On the other hand, as the accounts' balance approaches 0, the project is close to be liquidated and inside value of bonds should converge to their liquidation value  $\Phi$ . The interaction between the two effects results in a valley-shaped inside value near the

liquidation boundary, which is more significant when  $\rho$  is larger given that the term  $-\frac{m-r}{\sigma}\rho\psi$  in (36) makes coupon payment less.

Though it is not plotted, we also find that inside leverage, defined by  $\frac{D}{P}$ , surges if the entrepreneurial venture experiences a succession of poor performance; similar result is discovered in Biais et al. (2007).

### 4.3 Market Prices of Securities

Although the securities in the entrepreneurial venture are not traded in the market, the venture capital can still evaluate the market prices of the securities. To obtain the market prices of the securities, we follow the equilibrium asset pricing approach utilized by Goetzmann, Ingersoll, and Ross (2003). The crux of the equilibrium asset pricing approach contends that the market price of any cum-dividend claim discounted by the market martingale pricing operator must follow a martingale process under the physical probability measure in the absence of arbitrage.

Assume that the parallel complete market embedded in our model environment is epitomized by one wherein a representative investor maximizes his utility by selecting consumption and trading the market portfolio as well as the risk-free bond. It then follows that the unique market martingale pricing operator denoted by  $\Theta$  in this complete market evolves as

$$d\Theta_t = -r\Theta_t dt - \frac{m-r}{\sigma}\Theta_t dB_t. \quad (42)$$

Denote by  $\tilde{P}$  the market price of stocks in the entrepreneurial venture. Then given the optimal contract implementation,  $\Theta_t \tilde{P}_t + \int_t^\tau \Theta_s dL_s$  is a martingale under the physical measure from the standpoint of the market. By calculation of Itô's formula, we can derive the ordinary differential equation for  $\tilde{P}$  as

$$\begin{aligned} r\tilde{P}(y) = & \left[ ry + \frac{r\gamma}{2}\psi^2(1-\rho^2)(Z^*(y))^2 + h(A^*(Z^*(y))) \right] \tilde{P}'(y) \\ & + \frac{1}{2} \left[ \sigma^2(\zeta^*(y))^2 + (1-\rho^2)\psi^2(Z^*(y))^2 \right] \tilde{P}''(y) \end{aligned} \quad (43)$$

in the continuation region and with boundary conditions

$$\tilde{P}(0) = \max\{0, \Phi - F\}, \quad \text{and} \quad \tilde{P}'(\bar{y}) = \frac{1}{Z^*(\bar{y})}. \quad (44)$$

Similarly, denote by  $\tilde{D}$  the market price of bonds in the entrepreneurial venture. Then  $\Theta_t \tilde{D}_t + \int_t^\tau \Theta_s x_s ds$  is a martingale and satisfies the following ordinary differential equation,

$$\begin{aligned} r\tilde{D}(y) = & x(y) + \left[ ry + \frac{r\gamma}{2}\psi^2(1-\rho^2)(Z^*(y))^2 + h(A^*(Z^*(y))) \right] \tilde{D}'(y) \\ & + \frac{1}{2} \left[ \sigma^2(\zeta^*(y))^2 + (1-\rho^2)\psi^2(Z^*(y))^2 \right] \tilde{D}''(y) \end{aligned} \quad (45)$$

in the continuation region and with boundary conditions

$$\tilde{D}(0) = \min\{\Phi, F\}, \quad \text{and} \quad \tilde{D}'(\bar{y}) = 0. \quad (46)$$

We first solve the market price of stocks  $\tilde{P}$  under three various values of  $\rho$  in Panel A of Figure 5. Overall, dynamics of market price of stocks display similar pattern to dynamics of inside value. Note that the discounting factor used in market deviates from the discount rate  $\delta$  used by venture capital, because the expected return derived by the market is different from that derived by the venture capital. If we reclaim the drift of the manager's contract value in (43) by adding  $(m-r)\zeta^*(y)\tilde{P}'(y)$  back on both sides, we can observe that the actual expected return utilized by the market to discount market price of the private equity is given by

$$r + \frac{m-r}{\sigma}\sigma\zeta^*(y)\frac{\tilde{P}'(y)}{\tilde{P}(y)},$$

where the second term captures risk premium required by the market to price the stocks in the entrepreneurial venture. To comprehend this, first note that the dynamics of stock price are driven by the dynamics of risk management accounts in the entrepreneurial venture. Especially, from the market's perspective, only the marketwide shocks matter. According to  $Y$  in (21) with  $Z = Z^*$  and  $U = U^*$ , given  $Y = y$ , one unit marketwide shock will cause a change of  $\sigma\zeta^*(y)$  unit in the first risk management account. Next, market price of stocks will react to the change in the first account by a return-like multiplier  $\frac{\tilde{P}'(y)}{\tilde{P}(y)}$ , which is then multiplied by the market risk price  $\frac{m-r}{\sigma}$ . The ingredient  $\sigma\zeta^*(y)\frac{\tilde{P}'(y)}{\tilde{P}(y)}$  thus imitates market beta in a traditional CAPM equilibrium market. If  $\sigma\zeta^*\frac{\tilde{P}'(y)}{\tilde{P}(y)}$  is higher, the market expected return will be larger for pricing the stocks in the entrepreneurial venture, resulting in lower market price of the stocks. By referring to dynamics of  $\zeta^*$  in Panel B of Figure 1, we know that  $\zeta^*$  is positive near the liquidation boundary but turns to negative near the payment boundary. Therefore, market discount rate for pricing stocks will be lower than  $\delta$  when the balance of risk management accounts becomes sufficiently high. This explains why inside value of stocks is higher than market price of stocks but the difference narrows when the account balance increases.

Figure 5 about here

The characteristics of private debt's market price, on the other hand, are also notably influenced by market discount rate and hence diverge from those of the debt's inside value as is shown in Panel B of Figure 5. Since the market discounts at a higher rate than the venture capital due to an adjustment of market beta, the bonds' market price is lower than their corresponding inside

value. Finally, if the project is liquidated, market price of bonds converges to the liquidation value  $\Phi$ . Distinct from Ou-Yang (2005), our model implies that managerial incentives affect expected return of securities even with risk neutral principal. Characteristics of the entrepreneurial venture can influence market prices of securities via the channel of beta.

## 5 Conclusion

We have developed a dynamic agency model where a CARA manager who can perform unobservable managerial hedging seeks financing from a risk-neutral venture capital (principal) in order to initiate a project. In particular, the manager is protected by limited liability which can lead to inefficient liquidation of the project. The project inefficient liquidation causes the valuation function of the principal to become concave. Therefore the principal effectively behaves as if he were risk averse and optimally seeks risk-sharing with the manager. The resulting optimal contract is a mixture of relative performance evaluation and absolute performance evaluation. This provides one possible explanation of the mixed evidence on the use of relative performance evaluations. Moreover, the sensitivity of the optimal contract on the market return can become positive near project liquidation boundary. Consequently, a negative shock to the market can increase the likelihood of project liquidation. This is consistent with the evidence of heightened managerial turnover in bad market conditions.

The optimal contract is implemented by risk management accounts, private debt, and private equity. To award incentives so as to stimulate the unobservable managerial effort, two risk management accounts are necessary to track the state of the entrepreneurial venture, and they play the role of buffers in absorbing operation losses (or profits), as well as fulfilling the venture capital's risk-sharing objective. Discrepancy in discount rates between market and the venture capital explains the difference between market prices and inside values of these securities. In brief, comprehensive analyses show that the agent will acquit herself in a more conservative way by working more diligently under our optimal contract and that the principal is able to attain higher value with the optimal contract than with a pure relative or absolute performance evaluation contract.

Methodologically, CARA utility, monetary effort cost, and arithmetic Brownian motion of cash flow process are crucial assumptions breeding technical tractability and feasibility in our model. Consequentially, private saving and hedging, agent's risk aversion, and liquidation boundary are compatible in our unique model setup.

Given the compatibility and flexibility, our model can be extended in a number of directions. We could have followed previous works like Demarzo and Sannikov (2006), Sannikov (2008), and He

(2009) to extend our model by setting various pertinent boundary conditions. For instance, if the manager enjoyed outside options of career development, her quitting boundary would be shifted to the right. Costly replacement of the manager can be considered in our model. Since the principal's value function is increasing near the liquidation boundary, our contract is not renegotiation-proof, but can be revised to reach a renegotiation-proof one. A nontrivial possible extension is to consider ambiguity-proof contract as in Miao and Rivera (2016). Another interesting and important topic worth studying in the future, though potentially difficult, is to examine the effect of unobservable managerial hedging on the optimal contract if the agent is responsible for selecting risk components of the project without the principal's knowledge, for example, by following the setup in Cvitanić, Possamaï and Touzi (2017). Intuitively, we conjecture that a hedging manager would be predisposed not to choosing a project which is heavily loaded on idiosyncratic risk.

## 6 Appendix

### 6.1 A Model with Saving but without Managerial Hedging

For purpose of comparison, we inherit all the model environment from Section 2 to build up a parallel model with private saving but without managerial hedging. Therefore the agent has no access to trading the market portfolio, i.e.  $\pi = 0$ . Notice that even in this case, our model differs from that in He (2011) due to the agent's limited liability:  $dI \geq 0$ .

The agent's control problem is:

$$-\frac{1}{\gamma}e^{-\gamma(rS_t - j + rY_t)} = \Upsilon_t = \text{esssup}_{A,c} \mathbb{E}_t \left[ \bar{\delta} \int_t^\tau e^{-\bar{\delta}(s-t)} u(c_s) ds + e^{-\bar{\delta}(\tau-t)} u(rS_\tau - j) \right], \quad (47)$$

subject to the dynamics of private wealth process  $S$ :

$$dS_t = rS_t dt + dI_t - h(A_t)dt - c_t dt, \quad S_0 = \hat{s}, \quad (48)$$

the incentive compatibility constraint that  $A \in \mathcal{A}$  is the optimal effort for (47), and the participation constraint that  $Y \geq 0$ . The project is liquidated at  $\tau = \inf\{t \geq 0 : Y_t \leq 0\}$  when the agent quits. The constant  $j$  in (47) is defined by

$$j = \frac{1}{\gamma} \left( 1 - \frac{\bar{\delta}}{r} - \ln \left( \frac{r}{\bar{\delta}} \right) \right). \quad (49)$$

The agent's contract value is again denoted by  $Y$ . Address this problem by the same method as in Section 2, and we can show that the agent's optimal strategies are still given by (18) with  $\ell$  substituted by  $j$  and (19). The dynamics of  $Y$  follows

$$\begin{aligned} dY_t = & \left\{ rY_t + \frac{r\gamma}{2} \left[ \sigma^2 \zeta^2 + (1 - \rho^2) \psi^2 Z_t^2 \right] + h(A_t^*) \right\} dt \\ & - dI_t + \sigma \zeta dB_t + Z_t \sqrt{1 - \rho^2} \psi dB_t^\perp, \end{aligned} \quad (50)$$

where

$$\zeta = \frac{\rho\psi}{\sigma} Z + U.$$

The principal's control problem still remains the same as (24). Given (50), we can obtain the Hamilton-Jacobi-Bellman equation associated with the principal's control problem in the continuation region  $y \in (0, \bar{y})$  wherein  $dI = 0$  as

$$\begin{aligned} \delta W(y) = & \sup_{z \in \mathcal{Z}, \zeta \in \Gamma} \left\{ \delta(\mu + A^*(z)) + \left[ ry + \frac{r\gamma}{2} \sigma^2 \zeta^2 + \frac{r\gamma}{2} (1 - \rho^2) \psi^2 z^2 + h(A^*(z)) \right] W'(y) \right. \\ & \left. + \frac{1}{2} \left[ \sigma^2 \zeta^2 + (1 - \rho^2) \psi^2 z^2 \right] W''(y) \right\}, \end{aligned} \quad (51)$$

with three boundary conditions:

$$W(0) = \delta\Phi, \quad W'(\bar{y}) = -\delta, \quad \text{and} \quad W''(\bar{y}) = 0.$$

On the right-hand side of (51), terms including  $\zeta$  are

$$f(y, \zeta) = \frac{\sigma^2}{2} \left[ r\gamma W'(y) + W''(y) \right] \zeta^2.$$

If  $\zeta$  is allowed to take unbounded values (i.e.,  $\Gamma$  is an unbounded set), in order to have a finite value for  $W(y)$ ,  $\sup_{\zeta \in \Gamma} f(y, \zeta)$  must be finite, then, necessarily,  $r\gamma W'(y) + W''(y) \leq 0$ , which implies  $\sup_{\zeta} f(y, \zeta) = 0$  and  $\zeta^* = 0$ . Therefore we obtain the following result for the optimal contract.

**Proposition 6.1** *In the model with private saving but without managerial hedging, when the gross market exposure can be unbounded, the optimal contract is RPE.*

## 6.2 A Model with Absolute Performance Evaluation

In this section, we inherit all the model environment from Section 2 and solve models where the principal uses a pure APE or RPE contract. In either case, the agent's optimal market position, consumption rate, and effort are obtained by (17), (18), and (19), respectively given a contract  $(Y, Z, U, I)$ .

When the principal only uses APE, we set  $U \equiv 0$  in (14) and thus  $\zeta = \frac{\rho\psi}{\sigma}Z$  in (21). The HJB equation associated with the principal's optimization problem is

$$\begin{aligned} \delta W(y) = \sup_{z \in \mathcal{Z}} \left\{ \delta(\mu + A^*(z)) + \left[ ry + \frac{r\gamma}{2}\psi^2(1 - \rho^2)z^2 + h(A^*(z)) + \frac{(m-r)\rho\psi}{\sigma}z \right] W'(y) \right. \\ \left. + \frac{1}{2}\psi^2 z^2 W''(y) \right\}, \end{aligned}$$

in the continuation region  $(0, \hat{y})$  with boundary conditions

$$W(0) = \delta\Phi, \quad W'(\hat{y}) = -\delta, \quad \text{and} \quad W''(\hat{y}) = 0.$$

Figure A1 plots the numerical solution to the model with APE under three different values of  $\rho$ . In Panel A, the principal's value function uniformly *decreases* with respect to  $\rho$ , which is caused by wedge  $\frac{m-r}{\sigma}\rho\psi z W'(y) + \frac{1}{2}\rho^2\psi^2 z^2 W''(y)$ , since market return is not optimally loaded on the agent's contract value from the principal's point of view. High  $\rho$  and positive market risk price  $\frac{m-r}{\sigma}$  cause sufficient loss in value which outweighs the benefit of lower information asymmetry. Panel B shows that agent's optimal effort is not sensitive to  $\rho$ . When the contract value is near the liquidation boundary, the effort is binding at zero.

Figure A1 about here

### 6.3 A Model with Relative Performance Evaluation

When the principal only uses RPE, we set  $U = -\frac{\rho\psi}{\sigma}Z$  in (14) and thus  $\zeta \equiv 0$  in (21). The HJB equation associated with the principal's optimization problem is

$$\delta W(y) = \sup_{z \in \mathcal{Z}} \left\{ \delta(\mu + A^*(z)) + \left[ ry + \frac{r\gamma}{2}\psi^2(1 - \rho^2)z^2 + h(A^*(z)) \right] W'(y) + \frac{1}{2}\psi^2(1 - \rho^2)z^2 W''(y) \right\},$$

in the continuation region  $(0, \bar{y})$  with boundary conditions

$$W(0) = \delta\Phi, \quad W'(\bar{y}) = -\delta, \quad \text{and} \quad W''(\bar{y}) = 0.$$

Figure A2 plots the numerical solution to the model with RPE under three different values of  $\rho$ . Panel A shows that, in contrast to APE case, the principal's value function uniformly increases with respect to  $\rho$ . To understand the reason, recall that the principal completely filters out the marketwide shocks from the output. When  $\rho$  is higher, the degree of information asymmetry or agency friction is lower, and hence the principal's welfare is higher. Moreover, the RPE contract is more efficient to motivate the agent to exert higher effort as is demonstrated by Panel B.

Figure A2 about here

### 6.4 A Model with Contractible Managerial Investment

If the agent's investment in market portfolio  $\pi$  is contractible, then the principal can prescribe the agent's portfolio choice without respect to (17). Assume that the agent can still trade the market portfolio freely after quitting, then her control problem is

$$-\frac{1}{\gamma}e^{-\gamma(rS_t - \ell + rY_t)} = \Upsilon_t = \text{esssup}_{A, c} \mathbb{E}_t \left[ \bar{\delta} \int_t^\tau e^{-\bar{\delta}(s-t)} u(c_s) ds + e^{-\bar{\delta}(\tau-t)} u(rS_\tau - \ell) \right], \quad (52)$$

subject to the dynamics of private wealth process  $S$  in (4), the incentive compatibility constraint that  $A \in \mathcal{A}$  and  $c$  are maximizers of (52), and the participation constraint that  $Y \geq 0$ . The project is liquidated at  $\tau = \inf\{t \geq 0 : Y_t \leq 0\}$  when the agent retires. Address this problem by the same methodology in Section 2, and we can show that the agent's optimal strategies are still given by (18) and (19). The dynamics of  $Y$  are calculated as

$$\begin{aligned} dY_t = & \left[ rY_t + \frac{r\gamma}{2}\psi^2(1 - \rho^2)Z_t^2 + h(A_t^*) + (m - r)\zeta_t + \frac{r\gamma}{2}\sigma^2(\pi_t - \pi^*(\zeta_t))^2 \right] dt - dI_t \\ & + \zeta_t \sigma dB_t + Z_t \sqrt{1 - \rho^2} \psi dB_t^\perp, \end{aligned} \quad (53)$$

where  $\pi^*(\zeta) = \frac{m-r}{r\gamma\sigma^2} - \zeta$ .



Now the principal has control over  $\pi \in \Pi$  in addition to  $Z$ ,  $\zeta$ , and  $I$ , where  $\Pi$  is a compact set on real line. Given (53), we can obtain the HJB equation in the continuation region  $y \in (0, \tilde{y})$  as

$$\begin{aligned} \delta W(y) = & \sup_{z \in \mathcal{Z}, \zeta \in \Gamma, \pi \in \Pi} \left\{ \delta(\mu + A^*(z)) + \left[ ry + \frac{r\gamma}{2} \psi^2(1 - \rho^2) z^2 + h(A^*(z)) \right. \right. \\ & \left. \left. + \frac{r\gamma}{2} \sigma^2 (\pi - \pi^*(\zeta))^2 + (m - r)\zeta \right] W'(y) + \frac{1}{2} \left[ \sigma^2 \zeta^2 + (1 - \rho^2) \psi^2 z^2 \right] W''(y) \right\}. \end{aligned} \quad (54)$$

Similarly, there are three boundary conditions

$$W(0) = \mu, \quad W'(\tilde{y}) = -\delta, \quad \text{and} \quad W''(\tilde{y}) = 0.$$

It is clear in (54) that the term  $-\frac{r\gamma}{2} \sigma^2 (\pi - \pi^*(\zeta))^2 W'(y)$  stands for the cost of imposing on the agent an investment strategy  $\pi$  which is different from her optimal hedging behavior in (17). If  $W'(y) \leq 0$ , the agent's hedging incentive is aligned with the interest of the principal. Thus the principal optimally selects  $\pi = \pi^*(\zeta)$ . However, when  $W'(y) > 0$ , deviating from  $\pi = \pi^*(\zeta)$  makes the principal better off, and the resulting optimal contract dominates the OPE contract solved in Proposition 2.3.

## 6.5 Proof of Lemma 2.2

For any admissible strategy  $(A, c, \pi)$ , consider the process

$$\tilde{\Upsilon}_t = \int_0^t e^{-\bar{\delta}s} u(c_s) ds + e^{-\bar{\delta}t} \Upsilon_t^{A, c, \pi, Z, U},$$

where  $\Upsilon^{A, c, \pi, Z, U}$ , defined via (22) with  $\Upsilon_0^{A, c, \pi, Z, U} < \infty$ , satisfies (23). The definition of the dynamics for  $Y$  ensures that the drift of  $\tilde{\Upsilon}$  is non-positive, hence  $\tilde{\Upsilon}$  is a local supermartingale. Let  $\{\tau_n : n = 1, 2, 3, \dots\}$  be a localization sequence for its local martingale part such that  $\lim_{n \rightarrow \infty} \tau_n = \infty$  a.s.. We have

$$\mathbb{E} \left[ \int_0^{\tau_n \wedge \tau} e^{-\bar{\delta}s} u(c_s) ds \right] + \mathbb{E} \left[ e^{-\bar{\delta}\tau_n \wedge \tau} \Upsilon_{\tau_n \wedge \tau}^{A, c, \pi, Z, U} \right] \leq \tilde{\Upsilon}_0 = \Upsilon_0^{A, c, \pi, Z, U} = -\frac{1}{\gamma} e^{-\gamma(r\hat{s} - \ell + rY_0)}. \quad (55)$$

The second term on the left-hand side can be decomposed as

$$\mathbb{E} \left[ e^{-\bar{\delta}\tau_n} \Upsilon_{\tau_n}^{A, c, \pi, Z, U} 1_{\{\tau_n \leq \tau\}} \right] + \mathbb{E} \left[ e^{-\bar{\delta}\tau} \Upsilon_{\tau}^{A, c, \pi, Z, U} 1_{\{\tau < \tau_n\}} \right].$$

Sending  $n \rightarrow \infty$ , and using monotone convergence theorem together with (23) and the fact that  $\Upsilon_{\tau}^{A, c, \pi, Z, U} = u(rS_{\tau} - \ell)$  when  $\tau < \infty$ , we obtain

$$\mathbb{E} \left[ \int_0^{\tau} e^{-\bar{\delta}s} u(c_s) ds + e^{-\bar{\delta}\tau} u(rS_{\tau} - \ell) \right] \leq -\frac{1}{\gamma} e^{-\gamma(r\hat{s} - \ell + rY_0)}.$$

For strategy  $(A^*, c^*, \pi^*)$ ,  $\tilde{\Upsilon}$  is a local martingale. Then the inequality in (55) is an equality. Sending  $\tau_n \rightarrow \infty$ , the optimality of  $(A^*, c^*, \pi^*)$  is confirmed. ■

## 6.6 Proof of Proposition 2.3

Denote by  $\mathbb{F}^{R,X}$  the augmented filtration generated by the market return  $R$  and the output process  $X$ . The filtration  $\mathbb{F}^{R,X}$  is assumed to satisfy the usual conditions of completeness and right continuity. Recall from Section 2 that agent's effort takes values in a compact set  $\mathcal{A}$ . We set  $\mathcal{A} = [\underline{A}, \bar{A}]$  for some constants  $\underline{A} \leq \bar{A}$ . The cost of effort is given by a cost function

$$h(A) = \frac{\kappa}{2}A^2 + bA, \quad A \in \mathcal{A},$$

for some  $\kappa > 0$  and  $b > 0$ . Define  $h^*(Z) = \min_{A \in \mathcal{A}} \{h(A) - ZA\}$ , where the minimizer is

$$A^*(Z) = \operatorname{argmin} h^*(Z) = \begin{cases} \underline{A}, & \text{if } Z \leq h'(\underline{A}); \\ \frac{Z-b}{\kappa}, & \text{if } Z \in [h'(\underline{A}), h'(\bar{A})]; \\ \bar{A}, & \text{if } Z \geq h'(\bar{A}). \end{cases} \quad (56)$$

The principal's control problem is presented in (24) with state variable  $Y$  following the dynamics (21). In this control problem, the principal controls a nondecreasing compensation process  $I$ , contract sensitivity  $Z$ , and gross market exposure  $\zeta$ , which are adapted to  $\mathbb{F}^{R,X}$ . We assume that  $Z$  takes values inside  $\mathcal{Z} = [\underline{Z}, \bar{Z}]$  and  $\zeta$  takes values inside  $\Gamma = [\underline{\Gamma}, \bar{\Gamma}]$ , for constants  $\underline{Z} \leq \bar{Z}$  and  $\underline{\Gamma} \leq \bar{\Gamma}$ . We call the principal's control  $Z$ ,  $\zeta$ , and  $I$  admissible if they are adapted to  $\mathbb{F}^{R,X}$  and satisfy the transversality condition

$$\lim_{\bar{\tau} \rightarrow \infty} \mathbb{E} \left[ e^{-\delta \bar{\tau}} Y_{\bar{\tau}} 1_{\{\bar{\tau} \leq \tau\}} \right] = 0. \quad (57)$$

For given control variables  $Z$  and  $\zeta$ , we define

$$g(Z, \zeta) = \frac{r\gamma}{2} \psi^2 (1 - \rho^2) Z^2 + h(A^*(Z)) + (m - r)\zeta,$$

as *gross cost* which describes sum of the cost of exposing the agent to idiosyncratic risk and of effort and expected risk sharing return. We also denote

$$\Sigma(Z, \zeta) = (1 - \rho^2) \psi^2 Z^2 + \sigma^2 \zeta^2$$

as variance of the agent's contract value.

Throughout this section, we impose the following conditions on model parameters, the agent's and principal's control sets, and the agent's cost function.

### Assumption 6.2

(1)  $r > \delta$ ;

- (2)  $\sup_{Z \in \mathcal{Z}, \zeta \in \Gamma} \{A^*(Z) - g(Z, \zeta)\} \geq 0$ ;
- (3)  $g(Z^*, \zeta^*) \geq 0$  for  $(Z^*, \zeta^*) = \arg \max_{Z \in \mathcal{Z}, \zeta \in \Gamma} \{A^*(Z) - g(Z, \zeta)\}$ ;
- (4)  $\mathcal{A} = [0, \bar{A}]$  and  $\mathcal{Z} = [b, \kappa \bar{A} + b]$ ;
- (5)  $h(A) = \frac{\kappa}{2}A^2 + bA$  with  $\kappa > 0$  and  $b > 0$ ;
- (6)  $\delta\Phi \leq \mu$

Part (1) requires that the principal's discount rate  $\delta$  is less than the risk-free rate  $r$ . Part (2) assumes the existence of contract sensitivity and gross market exposure such that the expected present value of the cash flows that the principal can obtain from the agent's effort less the indirect cost to the principal is at least non-negative. Part (3) says that the gross cost is non-negative when the principal chooses the optimal sensitivity and market exposure to maximize the net contribution from the agent. The specification of  $\mathcal{Z}$  in Part (4) means that  $\mathcal{Z} = [h'(\underline{A}), h'(\bar{A})]$ . Therefore the principal's choice of contract sensitivity with respect to the project output coincides with agent's marginal cost of effort. If the principal chooses  $Z$  outside this range, it does not further motivate the agent to increase (resp. decrease) her effort to be higher than  $\bar{A}$  (resp. lower than  $\underline{A}$ ). The choice of strictly positive  $b$  in Part (5) implies that  $\mathcal{Z}$  is bounded away from zero, hence the principal's HJB equation (29) is uniformly elliptic. Finally, Part (6) states that the project's liquidation value to the principal is no more than its present value of expected output.

The problem (24) is a stochastic control problem with mixed singular control  $I$  and regular controls  $Z$  and  $\zeta$ . The principal's value function  $W$  is expected to satisfy the following HJB variational inequality

$$\min \left\{ \delta W - \sup_{Z \in \mathcal{Z}, \zeta \in \Gamma} \left\{ \delta(\mu + A^*(Z)) + (ry + g(Z, \zeta))W' + \frac{1}{2}\Sigma(Z, \zeta)W'' \right\}, W' + \delta \right\} = 0, \quad (58)$$

with the boundary condition  $W(0) = \delta\Phi$ .

Define

$$\bar{W}(y) = \mu - \delta y + \sup_{Z \in \mathcal{Z}, \zeta \in \Gamma} \{A^*(Z) - g(Z, \zeta)\} \quad \text{and} \quad \underline{W}(y) = \delta\Phi - \delta y, \quad \text{for } y \geq 0.$$

The following result characterizes  $W$  as a viscosity solution<sup>26</sup> to (58).

**Lemma 6.3** *Let Assumption 6.2 parts (1), (2), and (6) hold. Then  $\underline{W} \leq W \leq \bar{W}$  on  $[0, \infty)$ , and  $W$  is a unique continuous viscosity solution to (58).*

<sup>26</sup>See e.g., Pham (2009) Chapter 4 for the notion of viscosity solutions.

**Proof.** To prove  $W \leq \bar{W}$ , we first show that  $\bar{W}$  is a supersolution to (58). Indeed, since  $\bar{W}' = -\delta$  and  $\bar{W}'' = 0$ , then we have

$$\begin{aligned} & \delta \bar{W} - \sup_{Z \in \mathcal{Z}, \zeta \in \Gamma} \left\{ \delta(\mu + A^*(Z)) + (ry + g(Z, \zeta))\bar{W}' + \frac{1}{2}\Sigma(Z, \zeta)\bar{W}'' \right\} \\ &= \delta \bar{W} - \delta\mu + \delta ry - \delta \sup_{Z \in \mathcal{Z}, \zeta \in \Gamma} \{A^*(Z) - g(Z, \zeta)\} \\ &= \delta(r - \delta)y \\ &\geq 0, \end{aligned}$$

where the last inequality follows from  $r > \delta$  and  $y \geq 0$ . For the principal's any admissible control  $(Z, \zeta, I)$ , applying Itô's formula to  $\delta \int_0^{\tau \wedge T} e^{-\delta t} ((\mu + A^*(Z_t))dt - dI_t) + e^{-\delta(\tau \wedge T)} \bar{W}(Y_{\tau \wedge T})$  with  $Y_0 = y$  and using the assumption that  $Z$  and  $\zeta$  are bounded to ensure the stochastic integrals are martingales, we obtain from the supersolution property of  $\bar{W}$  that

$$\mathbb{E} \left[ \delta \int_0^{\tau \wedge T} e^{-\delta t} [(\mu + A^*(Z_t))dt - dI_t] + e^{-\delta(\tau \wedge T)} \bar{W}(Y_{\tau \wedge T}) \right] \leq \bar{W}(y), \quad \text{for any } T \geq 0.$$

Sending  $T \rightarrow \infty$ , using the fact that  $Y_\tau = 0$  when  $\tau < \infty$ , together with  $\bar{W}(0) \geq \mu \geq \delta\Phi$ , and the transversality condition (57), we obtain  $\mathbb{E} [\delta \int_0^\tau e^{-\delta t} ((\mu + A^*(Z_t))dt - dI_t) + \delta e^{-\delta\tau} \Phi] \leq \bar{W}(y)$ , which implies that  $W(y) \leq \bar{W}(y)$  since the choice of  $(Z, \zeta, I)$  is arbitrary.

For the lower bound of  $W$ , the principal can pay the agent a lump-sum transfer  $\Delta I_0 = y$  so that the agent quits immediately. This suboptimal strategy gives value  $\underline{W}(y) = \delta\Phi - \delta y \leq \mu - \delta y$ . Hence  $\underline{W}(y) \leq W(y)$ .

For the statement on viscosity solution, we turn to the dynamic programming principle. Define  $W^*(y) = \limsup_{\check{y} \rightarrow y} W(\check{y})$  and  $W_*(y) = \liminf_{\check{y} \rightarrow y} W(\check{y})$ , a standard argument using the dynamic programming principle (see e.g., Section 4.2 in Pham (2009)) shows that  $W^*$  is a (discontinuous) viscosity subsolution of (58) and  $W_*$  is a (discontinuous) viscosity supersolution. The comparison theorem<sup>27</sup> for viscosity solutions implies that  $W^* \leq W_*$ . But the reverse inequality  $W_* \leq W^*$  clearly holds due to their definitions. Therefore  $W_* = W^*$  which implies the continuity of  $W$ . Uniqueness then follows from the comparison theorem directly.  $\blacksquare$

In order to connect the HJB variational inequality (58) to the free boundary problem (29) with boundary conditions (26), (27), and (28), we introduce

$$\tilde{W}(y) = \mu - ry + \sup_{Z \in \mathcal{Z}, \zeta \in \Gamma} \{A^*(Z) - g(Z, \zeta)\}, \quad y \geq 0, \quad \text{and} \quad \bar{y} = \sup \left\{ y \geq 0 : \tilde{W}(y) > W(y) \right\}.$$

<sup>27</sup>The comparison theorem can be obtained by using Ishii's lemma. See Section 5 of Haussmann and Suo (1995) for a finite horizon stochastic control problem with mixed singular and regular control, and Chapter 5 of Xu (2017) for an infinite horizon case arising from a principal agent problem with reputation concern.

Since  $W \geq \underline{W}$ ,  $\bar{y}$  is at most the intersection of  $\tilde{W}$  and  $\underline{W}$ . Due to  $r > \delta$ , this intersection happens at a finite point, and hence  $\bar{y}$  is also finite. The continuity of  $W$  yields  $\tilde{W}(\bar{y}) = W(\bar{y})$ .

**Proposition 6.4** *Let Assumption 6.2 hold. Then  $W \in C^2(0, \infty)$  and satisfies*

$$\begin{aligned} \delta W &= \sup_{Z \in \mathcal{Z}, \zeta \in \Gamma} \left\{ \delta(\mu + A^*(Z)) + (ry + g(Z, \zeta))W' + \frac{1}{2}\Sigma(Z, \zeta)W'' \right\}, \quad y \in (0, \bar{y}); \\ W'(y) &= -\delta, \quad y \geq \bar{y}; \quad W(0) = \delta\Phi, \quad \text{and} \quad W''(\bar{y}) = 0. \end{aligned}$$

Moreover,  $W$  is concave on  $[0, \infty)$  and strictly concave on  $[0, \bar{y})$ .

**Proof.** The proof is separated into several steps.

**Step 1. Regularity and concavity on  $[0, \bar{y})$ .** Consider a boundary value problem

$$\delta V = \sup_{Z \in \mathcal{Z}, \zeta \in \Gamma} \left\{ \delta(\mu + A^*(Z)) + (ry + g(Z, \zeta))V' + \frac{1}{2}\Sigma(Z, \zeta)V'' \right\}, \quad y \in (0, \bar{y}), \quad (59)$$

with boundary condition  $V(0) = \delta\Phi$  and  $V(\bar{y}) = W(\bar{y})$ . Lemma 6.3 implies that  $W$  is a continuous viscosity solution of (59). On the other hand, Assumption 6.2 part (4) and (5) imply that (59) is uniformly elliptic, then Strulovici and Szydlowski (2015) Theorem 1<sup>28</sup> ensures that there exists  $V \in C^2(0, \bar{y})$  satisfying (59) and its boundary conditions. Since  $V$  is also a viscosity solution of (59), then the uniqueness of the viscosity solution (see e.g., Lions (1983)) implies that  $V \equiv W$ , hence  $W \in C^2(0, \bar{y})$ .

Since  $\Sigma(Z, \zeta)$  is bounded away from zero, we can rewrite (59) to obtain

$$W'' = \inf_{Z \in \mathcal{Z}, \zeta \in \Gamma} \left\{ \frac{\delta W - \delta(\mu + A^*(Z)) - (ry + g(Z, \zeta))W'}{\frac{1}{2}\Sigma(Z, \zeta)} \right\}, \quad y \in (0, \bar{y}). \quad (60)$$

Let  $(Z^*, \zeta^*) = \operatorname{argmax}_{Z \in \mathcal{Z}, \zeta \in \Gamma} \{A^*(Z) - g(Z, \zeta)\}$ . The right-hand side of the above equation is bounded from above by

$$\begin{aligned} & \frac{2}{\Sigma(Z, \zeta)} \left[ \delta W - \delta(\mu + A^*(Z^*)) - (ry + g(Z^*, \zeta^*))W' \right] \\ & \leq \frac{2}{\Sigma(Z, \zeta)} \left[ \delta \tilde{W} - \delta(\mu + A^*(Z^*)) + \delta(ry + g(Z^*, \zeta^*)) \right] \\ & = 0, \quad y \in (0, \bar{y}), \end{aligned}$$

where the inequality follows from  $W' \geq -\delta$ ,  $g(Z^*, \zeta^*) \geq 0$ , and  $y \geq 0$ . Besides, the inequality above is strict when  $y < \bar{y}$  since  $W(y) < \tilde{W}(y)$ . Combining the previous upper bound with (60) yields

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<sup>28</sup>The condition  $K_2^\mu < r$  in Strulovici and Szydlowski (2015) is not needed, since  $\mu + A^*(Z)$  is bounded in our setting; see Strulovici and Szydlowski (2015) footnote 21.

$W''(y) < 0$  when  $y < \bar{y}$ . Then the strict concavity of  $W$  and  $W' \geq -\delta$  implies that  $W'(y) > -\delta$  for all  $y \in (0, \bar{y})$ .

**Step 2.**  $W$  on  $(\bar{y}, \infty)$ . We will show that  $W(y) = -\delta(y - \bar{y}) + \tilde{W}(\bar{y})$  for all  $y \geq \bar{y}$ . To this end, define

$$V(y) = \begin{cases} W(y), & y \in [0, \bar{y}); \\ -\delta(y - \bar{y}) + W(\bar{y}), & y > \bar{y}. \end{cases}$$

In order to verify  $V \equiv W$ , it suffices to focus on  $[\bar{y}, \infty)$ . On one hand,  $W' \geq -\delta$  in viscosity sense; it follows from Touzi (2013) Lemma 5.22 that  $W(y) + \delta y$  is nondecreasing when  $y \geq \bar{y}$ . However  $V(y) + \delta y$  is a constant on  $[\bar{y}, \infty)$  and  $W(\bar{y}) = V(\bar{y})$ . Therefore  $V \leq W$  on  $[\bar{y}, \infty)$ . On the other hand, one can check that  $V$  is a supersolution of (58) on  $[\bar{y}, \infty)$ . Then the same argument for  $W \leq \bar{W}$  in Lemma 6.3 implies that  $W \leq V$  on  $[\bar{y}, \infty)$ . Therefore,  $V = W$  on  $[\bar{y}, \infty)$ , hence, on  $[0, \infty)$ . Clearly  $W$  is twice continuously differentiable on  $(\bar{y}, \infty)$  with  $W' = -\delta$ .

**Step 3.**  $W \in C^2(0, \infty)$ . Let us first prove  $W$  satisfies the smooth pasting condition, i.e.,  $W'(\bar{y}) = -\delta$ . The proof is similar to Pham (2009) Theorem 4.5.6. We have already seen from Step 1 that  $W'(y) > -\delta$  for  $y \in (0, \bar{y})$  and from Step 2 that  $W'(y) = -\delta$  for  $y > \bar{y}$ . Then  $W'_l(\bar{y}) \geq -\delta$  and  $W'_r(\bar{y}) = -\delta$ , where  $W'_l$  and  $W'_r$  are left and right derivatives respectively. Let us show that  $W'_l(\bar{y}) = -\delta$ . Assume otherwise that  $W'_l(\bar{y}) > -\delta$ . For any  $\eta \in (-\delta, W'_l(\bar{y}))$ , consider a test function  $\phi_n \in C^2(0, \infty)$  as:

$$\phi_n(y) = W(\bar{y}) + \eta(y - \bar{y}) - \frac{n}{2}(y - \bar{y})^2.$$

Then  $\phi'_n(\bar{y}) = \eta > -\delta$ ,  $\phi''_n(\bar{y}) = -n$ , and  $W - \phi_n$  attains the local maximum at  $\bar{y}$  with  $(W - \phi_n)(\bar{y}) = 0$ . Therefore, using  $\phi_n$  as a test function, the viscosity subsolution property of  $W$  implies that

$$\delta W(\bar{y}) - \sup_{Z \in \mathcal{Z}, \zeta \in \Gamma} \left\{ \delta(\mu + A^*(Z)) + (r\bar{y} + g(Z, \zeta))\eta - \frac{1}{2}\Sigma(Z, \zeta)n \right\} \leq 0.$$

Since  $\Sigma(Z, \zeta)$  is bounded away from zero, uniformly in  $Z$  and  $\zeta$ , by sending  $n \rightarrow \infty$  in the previous inequality, we get a contradiction.

Once  $W'(\bar{y}) = -\delta$ , sending  $y \uparrow \bar{y}$  in (60) and using  $W(\bar{y}) = \tilde{W}(\bar{y})$ , we obtain that the right-hand side of (60) converges to zero, hence  $W''_l(\bar{y}) = 0$ . Clearly  $W''_r(\bar{y}) = 0$ , therefore  $W \in C^2(0, \infty)$ . Thanks to the concavity of  $W$  on  $(0, \bar{y})$  and  $(\bar{y}, \infty)$ , and the condition of smooth pasting at  $\bar{y}$ ,  $W$  is globally concave on  $(0, \infty)$ . ■

In order to identify the principal's optimal controls, let us define

$$(Z^*(y), \zeta^*(y)) = \operatorname{argmax}_{Z \in \mathcal{Z}, \zeta \in \Gamma} \left\{ \delta A^*(Z) + g(Z, \zeta)W'(y) + \frac{1}{2}\Sigma(Z, \zeta)W''(y) \right\}.$$

Note that the first-order condition on  $\zeta$  yields

$$\zeta^*(y) = \max \left\{ \underline{\Gamma}, \min \left\{ \bar{\Gamma}, -\frac{m-r}{\sigma^2} \frac{W'(y)}{W''(y)} \right\} \right\}. \quad (61)$$

Let us prepare the following result.

**Lemma 6.5** *Let Assumption 6.2 parts (4) and (5) hold. Then  $Z^*$  and  $\zeta^*$  are locally Lipschitz on  $(0, \bar{y}]$ .*

**Proof.** For each  $n > 0$ , we need to show that  $Z^*$  and  $\zeta^*$  are Lipschitz on  $[\frac{1}{n}, \bar{y}]$ . Fix  $n$  and define

$$f(Z, \zeta, y) = \delta A^*(Z) + g(Z, \zeta)W'(y) + \frac{1}{2}\Sigma(Z, \zeta)W''(y), \quad \text{for } y \in [\frac{1}{n}, \bar{y}].$$

Let us first consider the first-order condition for  $Z$  in  $f$ , when  $Z$  is not necessarily constrained in  $\mathcal{Z}$ . Since  $h(A) = \frac{\kappa}{2}A^2 + bA$ , then  $A^*(Z) = (h')^{-1}(Z) = \frac{Z-b}{\kappa}$ . Calculation shows that

$$\partial_Z f = \frac{\delta}{\kappa} + Z \left[ \left( \frac{1}{\kappa} + r\gamma\psi^2(1-\rho^2) \right) W'(y) + (1-\rho^2)\psi^2 W''(y) \right].$$

When  $\left( \frac{1}{\kappa} + r\gamma\psi^2(1-\rho^2) \right) W'(y) + (1-\rho^2)\psi^2 W''(y) \neq 0$ , the solution to the first-order condition  $\partial_Z f = 0$  is

$$\hat{Z}(y) = -\frac{\delta}{\kappa} \frac{1}{\left( \frac{1}{\kappa} + r\gamma\psi^2(1-\rho^2) \right) W'(y) + (1-\rho^2)\psi^2 W''(y)}.$$

Constraining  $\hat{Z}(y)$  to  $\mathcal{Z}$ , we define  $Z^*(y) = \max \{ h'(\underline{A}), \min \{ h'(\bar{A}), \hat{Z}(y) \} \}$ . Consequently, the maximizer  $Z^*(y)$  is attained either at  $\hat{Z}(y)$ ,  $h'(\underline{A})$ , or  $h'(\bar{A})$ .

Let us consider the case where  $Z^*(y) = \hat{Z}(y)$ , but not the boundary points  $h'(\underline{A})$  or  $h'(\bar{A})$ . In this case, we claim that  $Z^*(y)$  is Lipschitz continuous when  $y \in [\frac{1}{n}, \bar{y}]$ . To prove this claim, we first observe that it is necessary to have  $\left( \frac{1}{\kappa} + r\gamma\psi^2(1-\rho^2) \right) W'(y) + (1-\rho^2)\psi^2 W''(y)$  bounded away from zero in order to have  $h'(\underline{A}) < \hat{Z}(y) < h'(\bar{A})$ . On the other hand, the twice continuous differentiability of  $W$  in Proposition 6.4 implies that  $W'$  and  $W''$  are both bounded in  $[\frac{1}{n}, \bar{y}]$ . Then taking another derivative with respect to  $y$  on both sides of (60) and applying the enveloping theorem, we obtain that  $W'''$  is bounded on  $[\frac{1}{n}, \bar{y}]$  as well. Therefore, when  $Z^*(y)$  is attained at  $\hat{Z}(y)$  and  $h'(\underline{A}) < \hat{Z}(y) < h'(\bar{A})$ , we have

$$(Z^*(y))' = -\frac{\delta}{\kappa} \frac{\left( \frac{1}{\kappa} + r\gamma\psi^2(1-\rho^2) \right) W''(y) + (1-\rho^2)\psi^2 W'''(y)}{\left[ \left( \frac{1}{\kappa} + r\gamma\psi^2(1-\rho^2) \right) W'(y) + (1-\rho^2)\psi^2 W''(y) \right]^2},$$

which is bounded on  $[\frac{1}{n}, \bar{y}]$ , because we have seen that the numerator is bounded and the denominator is bounded away from zero. Therefore we confirm the claim. Now combining the Lipschitz

continuity of  $Z^*(y)$  in  $(h'(\underline{A}), h'(\bar{A}))$  and the cases where  $Z^*(y)$  is equal to  $h'(\underline{A})$  or  $h'(\bar{A})$ , we prove that  $Z^*(y)$  is Lipschitz continuous when  $y \in [\frac{1}{n}, \bar{y}]$ .

To prove the claim on  $\zeta^*$ , we observe that  $W''(y)$  is necessarily bounded away from zero when  $\zeta^*(y)$  is attained at  $-\frac{m-r}{\sigma^2} \frac{W'(y)}{W''(y)}$ . Therefore, boundedness of  $W', W''$  and  $W'''$  on  $[\frac{1}{n}, \bar{y}]$  implies that  $\zeta^*(y)$  is Lipschitz continuous when it is attained at  $-\frac{m-r}{\sigma^2} \frac{W'(y)}{W''(y)}$ . Combining with the binding cases, we obtain the Lipschitz continuity of  $\zeta^*(y)$  in (61) when  $y \in [\frac{1}{n}, \bar{y}]$ . ■

Define  $U^*(y) = \zeta^*(y) - \frac{\rho\psi}{\sigma} Z^*(y)$  for  $y \in [0, \bar{y}]$ .

**Corollary 6.6** *Let Assumption 6.2 hold. For the agent's any admissible effort  $A = \{A_t : t \geq 0\}$ , there exists unique processes  $Y$  and  $I$  solving the following stochastic differential equation (SDE) with reflection*

$$\begin{aligned} dY_t = & [rY_t + g(Z^*(Y_t), \zeta^*(Y_t)) - (\mu + A_t)Z^*(Y_t) - mU^*(Y_t)]dt \\ & + Z^*(Y_t)dX_t + U^*(Y_t)dR_t - dI_t, \quad 1_{\{Y_t < \bar{y}\}}dI_t = 0, \quad Y_t \in [0, \bar{y}], \end{aligned}$$

when  $0 \leq t \leq \tau$  and where  $\tau = \inf\{t \geq 0 : Y_t \leq 0\}$ . When  $A = A^*(Z^*)$ ,  $Y$  and  $I$  are both  $\mathbb{F}^{R,X}$ -adapted.

**Proof.** The above SDE with reflection can be rewritten as

$$\begin{aligned} dY_t = & (rY_t + g(Z^*(Y_t), \zeta^*(Y_t)))dt + \zeta^*(Y_t)\sigma dB_t + Z^*(Y_t)\sqrt{1 - \rho^2}\psi dB_t^\perp - dI_t, \\ & 1_{\{Y_t < \bar{y}\}}dI_t = 0, \quad Y_t \in [0, \bar{y}], \quad \text{when } 0 \leq t \leq \tau. \end{aligned}$$

For each  $n > 0$ , Lemma 6.5 shows that coefficients of this reflected SDE are Lipschitz when  $Y_t \in [\frac{1}{n}, \bar{y}]$ . It then follows from Tanaka (1979) that this reflected SDE admits a unique strong solution until  $\tau_n = \inf\{t \geq 0 : Y_t \leq \frac{1}{n}\}$ . Sending  $n \rightarrow \infty$ , the solution can be uniquely extended to  $\tau$ .

When  $A = A^*(Z^*)$ , consider the following SDE without reflection

$$d\tilde{Y}_t = \left[ r\tilde{Y}_t + g(Z^*(\tilde{Y}_t)) - (\mu + A^*(Z^*(\tilde{Y}_t)))Z^*(\tilde{Y}_t) - mU^*(\tilde{Y}_t) \right]dt + Z^*(\tilde{Y}_t)dX_t + U^*(\tilde{Y}_t)dR_t,$$

and a local time term  $I_t = \tilde{Y}_t - \max_{0 \leq s \leq t} \max\{\tilde{Y}_s, \bar{y}\}$ . Then  $Y = \tilde{Y} - I$ . Since  $\tilde{Y}$  is a strong solution to the previous SDE with local Lipschitz continuous coefficients,  $\tilde{Y}$  is adapted to the filtration generated by  $R$  and  $X$ , and so is  $(Y, I)$ . ■

Now we are ready to prove Proposition 2.6.

**Proof of Proposition 2.6.** Let us denote  $Z_t^* = Z^*(Y_t)$ ,  $U_t^* = U^*(Y_t)$ , and  $A_t^* = A^*(Z_t^*)$ . Recall  $(\pi^*, c^*)$  from (13) and (14), where  $Z$  and  $U$  are replaced by  $Z^*$  and  $U^*$  respectively. We will show



the strategy  $(A^*, c^*, \pi^*)$  is admissible. When the agent employs this strategy, we have seen from (34) that

$$d\left(S_t - \frac{\ell}{r} + Y_t\right) = \left[\frac{(m-r)^2}{2r\gamma\sigma^2} + \frac{1}{\gamma}\left(1 - \frac{\bar{\delta}}{r}\right) + \frac{r\gamma}{2}\psi^2(1 - \rho^2)(Z_t^*)^2\right] dt + \frac{m-r}{r\gamma\sigma} dB_t + Z_t^* \psi \sqrt{1 - \rho^2} dB_t^\perp.$$

Then

$$e^{-\gamma(rS_t - \ell + rY_t)} = e^{-\gamma(rS_0 - \ell + rY_0)} e^{(\bar{\delta} - r)t} \mathcal{E}\left(-\int_0^t \frac{m-r}{\sigma} dB_s - \int_0^t r\gamma Z_s^* \psi \sqrt{1 - \rho^2} dB_s^\perp\right),$$

where  $\mathcal{E}\left(\int_0^t \alpha_s dB_s\right)$  is the stochastic exponential of  $\int_0^t \alpha_s dB_s$ . The stochastic exponential above is a positive local martingale, hence, a supermartingale. Using the bounded convergence and  $r > 0$ , we can verify that the transversality condition (19) is satisfied. Therefore  $(A^*, c^*, \pi^*)$  is admissible to the agent. It then follows from Lemma 2.2 that  $A^*$  is agent's optimal effort.

For the principal, since  $Y$  is bounded, the transversality condition (57) is clearly satisfied. Then Corollary 6.6 implies that  $(Z^*, U^*, I)$  is admissible to the principal. A standard verification argument, similar to Proposition 1 in DeMarzo and Sannikov (2006), confirms that  $(Z^*, U^*, I)$  is optimal for the principal.  $\blacksquare$

## 6.7 Details for the Numerical Method

We assume that  $A \in \mathcal{A} = [\underline{A}, \bar{A}]$ , where  $\underline{A}$  is set at 0 so that we exclude negative effort. Then we calculate the agent's optimal effort choice in (19) according to (56).

To determine the lower bound of  $\zeta$ , note that in the capital structure implementation,  $U$  stands for the position of market portfolio in the first risk management account. When establishing the accounts, the principal is required to collateralize the asset for short-selling the market portfolio. Thus, we assume that the principal faces short-selling restriction: total short position in the entrepreneurial venture's risk management accounts should be bounded from below at least by the liquidation value of the collateralized asset, i.e.

$$\frac{U^*}{Z^*} \geq -\Phi, \quad \text{for all } y \in (0, \bar{y}]. \quad (62)$$

Since  $U^* = \zeta^* - \frac{\rho\psi}{\sigma} Z^*$ , we select  $\bar{\Gamma}$  so that (62) is satisfied in the numerical experiments.

Under Assumption 6.2 in the previous section, we can numerically solve the principal's control problem (29). To this end, we rearrange the equation to

$$W''(y) = \inf_{z \in [\underline{Z}, \bar{Z}], \zeta \in [\underline{\Gamma}, \bar{\Gamma}]} \left\{ \frac{2\delta}{\sigma^2 \zeta^2 + (1 - \rho^2) \psi^2 z^2} \left( W(y) - \mu - \frac{z-b}{\kappa} \right) - \frac{2}{\sigma^2 \zeta^2 + (1 - \rho^2) \psi^2 z^2} \left[ ry + \frac{r\gamma}{2} \psi^2 (1 - \rho^2) z^2 + \frac{z^2 - b^2}{2\kappa} + (m - r)\zeta \right] W'(y) \right\}. \quad (63)$$

First assume that  $z(y)$  and  $\zeta(y)$  are interior solutions given  $y$ , the first-order condition for  $z$  with respect to (63) is

$$\begin{aligned} & \left[ -\frac{\delta}{\kappa} - (r\gamma\psi^2(1-\rho^2) + \frac{1}{\kappa}) W'(y)z \right] [\sigma^2\zeta^2 + (1-\rho^2)\psi^2z^2] - 2(1-\rho^2)\psi^2z \left\{ \delta W(y) \right. \\ & \left. - \delta\left(\mu + \frac{z-b}{\kappa}\right) - \left[ ry + \frac{r\gamma}{2}\psi^2(1-\rho^2)z^2 + \frac{z^2-b^2}{2\kappa} + (m-r)\zeta \right] W'(y) \right\} = 0, \end{aligned} \quad (64)$$

and for  $\zeta$  is

$$\begin{aligned} & -(m-r)W'(y) [\sigma^2\zeta^2 + (1-\rho^2)\psi^2z^2] - 2\sigma^2\zeta \left\{ \delta W(y) - \delta\left(\mu + \frac{z-b}{\kappa}\right) \right. \\ & \left. - \left[ ry + \frac{r\gamma}{2}\psi^2(1-\rho^2)z^2 + \frac{z^2-b^2}{2\kappa} + (m-r)\zeta \right] W'(y) \right\} = 0. \end{aligned} \quad (65)$$

It can be derived by algebraic manipulation that

$$(m-r)W'(y)\zeta + \frac{\delta}{\kappa}z = 2 \left[ \delta W(y) - \delta\left(\mu - \frac{b}{\kappa}\right) - \left(ry - \frac{b^2}{2\kappa}\right) W'(y) \right], \quad (66)$$

and

$$\zeta = \frac{(m-r)(1-\rho^2)\psi^2W'(y)z}{\sigma^2 \left[ \frac{\delta}{\kappa} + \left(\frac{1}{\kappa} + r\gamma\psi^2(1-\rho^2)\right) W'(y)z \right]}. \quad (67)$$

Combining the above two equations, we can show that  $z$  solves a quadratic equation  $\mathbb{A}z^2 + \mathbb{B}z + \mathbb{C} = 0$ , where we define

$$\begin{aligned} \mathbb{A} &= \frac{\delta}{2\kappa} \left( \frac{1}{\kappa} + r\gamma\psi^2(1-\rho^2) \right) W'(y), \\ \mathbb{B} &= (1-\rho^2)\psi^2 \frac{(m-r)^2}{2\sigma^2} (W'(y))^2 + \frac{\delta^2}{2\kappa^2} \\ &\quad - \left( \frac{1}{\kappa} + r\gamma\psi^2(1-\rho^2) \right) \left[ \delta W(y) - \delta\left(\mu - \frac{b}{\kappa}\right) - \left(ry - \frac{b^2}{2\kappa}\right) W'(y) \right] W'(y), \\ \mathbb{C} &= -\frac{\delta}{\kappa} \left[ \delta W(y) - \delta\left(\mu - \frac{b}{\kappa}\right) - \left(ry - \frac{b^2}{2\kappa}\right) W'(y) \right]. \end{aligned}$$

Since  $z$  must conform to the second-order condition and be continuous at  $\mathbb{A} = 0$  where  $W'(y) = 0$ ,  $z$  should be the larger root, that is, we have

$$z = -\frac{2\mathbb{C}}{\sqrt{\mathbb{B}^2 - 4\mathbb{A}\mathbb{C}} + \mathbb{B}}. \quad (68)$$

Substitute  $z$  into (67), and we can obtain  $(z, \zeta)$ . If  $z \in [Z, \bar{Z}]$  and  $\zeta \in [\Gamma, \bar{\Gamma}]$ , then we have  $Z^* = z$  and  $\zeta^* = U^* + \frac{\rho\psi}{\sigma} Z^* = \zeta$ .

If only  $z$  is binding at either boundary, then  $Z^*$  takes the closest boundary value. And we substitute the binding boundary value of  $z$  into the first-order condition (65) to update  $\zeta$ . Likewise, if only  $\zeta$  is binding at its boundary, we use (64) to recalculate  $z$ . If both  $z$  and  $\zeta$  are binding, then both take their boundary values respectively. We analytically and numerically check the second-order conditions for  $Z^*$  and  $\zeta^*$  on their ranges and verify the global optimality of them.

We use bisection-shooting method to solve equation (63) on a grid of  $y$  beginning at 0 with length of step  $\Delta y$ . Given  $W(0) = \mu$ , we conjecture an initial first order derivative  $W'(0) = \omega$ . With this initial condition, the boundary value problem is transferred into an initial value problem: given  $W$  and  $W'$  at  $y$ , we can calculate the optimizers  $(z(y), \zeta(y))$  as above, substitute them into (63) and obtain the second order derivative  $W''(y)$ . The value of functions  $W$  and  $W'$  in the next step could be calculated by  $W(y + \Delta y) = W(y) + W'(y)\Delta y$  and  $W'(y + \Delta y) = W'(y) + W''(y)\Delta y$ . Then we can calculate  $(z, \zeta)$  and  $W''$  at the next step as above. The shooting is terminated either we have  $W' + \delta = 0$  or  $W'' = 0$ . If  $W' + \delta = 0$  happens before  $W'' = 0$ , we increase  $\omega$ ; otherwise, we decrease  $\omega$ . When the algorithm converges, we can identify the unique initial first order derivative to pin-down the function  $W$ .

## 6.8 Proof of Proposition 4.1

It suffices to show that  $Y$  follows dynamics (21) with  $Z = Z^*$  and  $U = U^*$  under the capital structure implementation. Given the project is not liquidated, i.e.,  $Y_t \in (0, \bar{y}]$ , define  $M_t = \int_0^t \frac{1}{Z_s^*} dY_s$  as the total balance of risk management accounts at  $t$ . Note that the ratio of two accounts is always equal to  $\frac{Z^*}{1-Z^*}$ , and that total position in the market portfolio is  $\frac{U^*}{Z^*}$ , so the resource constraint in the above capital structure is

$$dX_t + r \left( M_t - \frac{U_t^*}{Z_t^*} \right) dt + \frac{U_t^*}{Z_t^*} dR_t = dM_t + x_t dt + dL_t. \quad (69)$$

Substitute (36) and (37) into resource constraint (69), and we have

$$\begin{aligned} dM_t = & \left[ rM_t + \frac{r\gamma}{2}\psi^2(1-\rho^2)Z_t^* + \frac{h(A_t^*)}{Z_t^*} + \frac{m-r}{\sigma}\rho\psi + (m-r)\frac{U_t^*}{Z_t^*} \right] dt \\ & + \left( \rho\psi + \frac{U_t^*}{Z_t^*}\sigma \right) dB_t + \sqrt{1-\rho^2}\psi dB_t^\perp - \frac{dI_t}{Z_t^*(\bar{y})}. \end{aligned} \quad (70)$$

Multiplying  $dM_t$  by  $Z_t^*$ , and using the fact that  $Z_t^*dM_t = dY_t$  and  $dI_t > 0$  if and only if  $Y_t = \bar{y}$ , we verify that  $Y$  follows the dynamics (21) with  $Z = Z^*$  and  $U = U^*$ . ■

## 6.9 Details for Inside Values of Securities and Balance Sheet

In accordance with the approach introduced by Black and Scholes (1973), the dynamics of inside values of stocks and bonds can be represented by ordinary differential equations with appropriate boundary conditions.

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<sup>29</sup>The left-hand side of the resource constraint is the sum of cash flows generated by the project and investment return of the risk management accounts. The right-hand side of the resource constraint represents three outlays of cash flows: after coupon and dividend payment, surplus cash is deposited into the balance of risk management accounts.

The inside value of stocks in (38) satisfies

$$\begin{aligned} \delta P(y) = & \left[ ry + \frac{r\gamma}{2}\psi^2(1-\rho^2)(Z^*(y))^2 + h(A^*(Z^*(y))) + (m-r)\zeta^*(y) \right] P'(y) \\ & + \frac{1}{2} [\sigma^2(\zeta^*(y))^2 + (1-\rho^2)\psi^2(Z^*(y))^2] P''(y) \end{aligned} \quad (71)$$

in the continuation region and with boundary conditions

$$P(0) = \max \{0, \Phi - F\}, \quad (72)$$

and

$$P'(\bar{y}) = \frac{1}{Z^*(\bar{y})}. \quad (73)$$

Similarly, the inside value of bonds in (39) satisfies

$$\begin{aligned} \delta D(y) = & \mu - \frac{m-r}{\sigma}\rho\psi + A^*(Z^*(y)) - \frac{h(A^*(Z^*(y)))}{Z^*(y)} - \frac{r\gamma}{2}\psi^2(1-\rho^2)Z^*(y) \\ & + \left[ ry + \frac{r\gamma}{2}\psi^2(1-\rho^2)(Z^*(y))^2 + h(A^*(Z^*(y))) + (m-r)\zeta^*(y) \right] D'(y) \\ & + \frac{1}{2} [\sigma^2(\zeta^*(y))^2 + (1-\rho^2)\psi^2(Z^*(y))^2] D''(y) \end{aligned} \quad (74)$$

in the continuation region and with boundary conditions

$$D(0) = \min \{\Phi, F\}, \quad (75)$$

and

$$D'(\bar{y}) = 0. \quad (76)$$

The first boundary condition for inside value of stocks (72) describes the value of private equity that the venture capital would receive should the project be liquidated. Since such equity is residual claim on the project, the equity partners in the venture capital obtain positive value only if the liquidation value of the project  $\Phi$  exceeds face value of the debt  $F$ . The second boundary condition for inside value of stocks (73) specifies total dividend payment for both the venture capital and the manager. Similarly, the first boundary condition for inside value of bonds (75) states how much value the debt partners would procure in case of project liquidation: the upper bound of this amount of value is the face value  $F$ . The second boundary condition for inside value of bonds (76) means that the debt partners are not eligible for dividend payment.

## 6.10 Proof of Proposition 4.2

Define  $\mathcal{T} = \min\{T, \tau\}$  for a constant  $T \geq t$ . Rewrite stochastic differential equation  $\frac{d(e^{-\delta t} Y_t)}{Z_t^*} = -\delta e^{-\delta t} \frac{Y_t}{Z_t^*} dt + e^{-\delta t} \frac{dY_t}{Z_t^*}$  into an integral form

$$\begin{aligned} e^{-\delta(\mathcal{T}-t)} \frac{Y_{\mathcal{T}}}{Z_{\mathcal{T}}^*} &= \frac{Y_t}{Z_t^*} + \int_t^{\mathcal{T}} e^{-\delta(s-t)} \left[ \frac{r\gamma}{2} \psi^2 (1 - \rho^2) Z_s^* + \frac{h(A_s^*)}{Z_s^*} + \frac{(m-r)\rho\psi}{\sigma} + (m-r) \frac{U_s^*}{Z_s^*} \right] ds \\ &\quad + \int_t^{\mathcal{T}} e^{-\delta(s-t)} (r - \delta) \frac{Y_s}{Z_s^*} ds + \int_t^{\mathcal{T}} e^{-\delta(s-t)} \left( \frac{\rho\psi}{\sigma} + \frac{U_s^*}{Z_s^*} \right) dB_s \\ &\quad + \int_t^{\mathcal{T}} e^{-\delta(s-t)} \sqrt{1 - \rho^2} \psi dB_s^\perp - \int_t^{\mathcal{T}} e^{-\delta(s-t)} \frac{1}{Z_s^*} dI_s. \end{aligned}$$

Since  $U^*$  is bounded and  $Z^*$  is bounded away from zero (see Assumption 6.2 Part (4)), then both stochastic integrands on the right-hand side are bounded, hence both stochastic integrals are martingales. Take conditional expectations on both sides of the above equation and let  $T$  go to  $\infty$ . Because  $Y$  is bounded and  $Z^*$  is bounded away from zero, the bounded convergence theorem implies that  $\lim_{T \rightarrow \infty} \mathbb{E}_t \left[ e^{-\delta(\mathcal{T}-t)} \frac{Y_{\mathcal{T}}}{Z_{\mathcal{T}}^*} \right] = 0$ . So we have

$$\begin{aligned} \frac{Y_t}{Z_t^*} + Q_t &= -\mathbb{E}_t \left[ \int_t^{\tau} e^{-\delta(s-t)} \left[ \frac{r\gamma}{2} \psi^2 (1 - \rho^2) Z_s^* + \frac{h(A_s^*)}{Z_s^*} + \frac{(m-r)\rho\psi}{\sigma} \right] ds \right] \\ &\quad + \mathbb{E}_t \left[ \int_t^{\tau} e^{-\delta(s-t)} \frac{1}{Z_s^*} dI_s \right]. \end{aligned}$$

Dividing (24) by  $\delta$ , we have the value enjoyed by the venture capital as the present value of the project cash flows

$$\frac{W(Y_t)}{\delta} = \mathbb{E}_t \left[ \int_t^{\tau} e^{-\delta(s-t)} (\mu + A_s^*) ds \right] - \mathbb{E}_t \left[ \int_t^{\tau} e^{-\delta(s-t)} dI_s \right] + \mathbb{E}_t \left[ e^{-\delta(\tau-t)} \Phi \right]. \quad (77)$$

Summing up the above two equations, we have

$$\begin{aligned} \frac{W(Y_t)}{\delta} + \frac{Y_t}{Z_t^*} + Q_t &= \mathbb{E}_t \left[ \int_t^{\tau} e^{-\delta(s-t)} \left[ \mu + A_s^* - \frac{r\gamma}{2} \psi^2 (1 - \rho^2) Z_s^* - \frac{h(A_s^*)}{Z_s^*} - \frac{(m-r)\rho\psi}{\sigma} \right] ds \right] \\ &\quad + \mathbb{E}_t \left[ \int_t^{\tau} e^{-\delta(s-t)} \frac{1 - Z_s^*}{Z_s^*} dI_s \right] + \mathbb{E} \left[ e^{-\delta(\tau-t)} \Phi \right]. \end{aligned}$$

Since  $dI_s > 0$  if and only if  $Y_s = \bar{y}$ , we have  $\mathbb{E}_t \left[ \int_t^{\tau} e^{-\delta(s-t)} \frac{1 - Z_s^*}{Z_s^*} dI_s \right] = (1 - Z^*(\bar{y})) \mathbb{E}_t \left[ \int_t^{\tau} e^{-\delta(s-t)} dL_s \right]$ .

On the other hand,  $\Phi = \max\{0, \Phi - F\} + \min\{\Phi, F\}$ . Finally, refer to the definition of  $x$  in (36),

and it transpires that

$$\begin{aligned} \frac{W(Y_t)}{\delta} + \frac{Y_t}{Z^*(Y_t)} + Q_t &= \mathbb{E}_t \left[ \int_t^{\tau} e^{-\delta(s-t)} x_s ds \right] + \mathbb{E}_t \left[ e^{-\delta(\tau-t)} \min\{\Phi, F\} \right] \\ &\quad + (1 - Z^*(\bar{y})) \mathbb{E}_t \left[ \int_t^{\tau} e^{-\delta(s-t)} dL_s \right] + Z^*(\bar{y}) \mathbb{E}_t \left[ e^{-\delta(\tau-t)} \max\{0, \Phi - F\} \right] \\ &\quad + (1 - Z^*(\bar{y})) \mathbb{E}_t \left[ e^{-\delta(\tau-t)} \max\{0, \Phi - F\} \right] \\ &= D(y) + (1 - Z^*(\bar{y}))P(y) + Z^*(\bar{y}) \mathbb{E}_t \left[ e^{-\delta(\tau-t)} \max\{0, \Phi - F\} \right]. \quad \blacksquare \end{aligned}$$

## 6.11 Details for Market Prices of Securities

We derive the ordinary differential equation for market price of stocks here; that for market price of bonds can be obtained in similar way. Assume that at  $t \geq 0$ , the project is not liquidated. Itô's formula implies that the drift of  $\Theta_t \tilde{P}_t + \int_t^\tau \Theta_s dL_s$  (divided throughout by  $\Theta_t$ ) is

$$\begin{aligned} & \left\{ -r\tilde{P}(Y_t) + \left[ rY_t + \frac{r\gamma}{2}\psi^2(1-\rho^2)Z_t^{*2} + h(A_t^*) \right] \tilde{P}'(Y_t) \right. \\ & \left. + \frac{1}{2} \left[ (\rho\psi Z_s^* + \sigma U_s^*)^2 + (1-\rho^2)\psi^2 Z_s^{*2} \right] \tilde{P}''(Y_t) \right\} dt + \left( \frac{1}{Z^*(\bar{y})} - \tilde{P}'(Y_t) \right) dI_t. \end{aligned}$$

Since  $dI > 0$  if and only if  $Y_t = \bar{y}$ , when  $Y_t = y \in (0, \bar{y})$ , we must have (43) so that  $\Theta_t \tilde{P}_t + \int_t^\tau \Theta_s dL_s$  is a martingale. When  $Y_t = \bar{y}$ ,  $\tilde{P}'(Y_t) = \frac{1}{Z^*(\bar{y})}$  gives the boundary condition  $\tilde{P}'(\bar{y}) = \frac{1}{Z^*(\bar{y})}$ .

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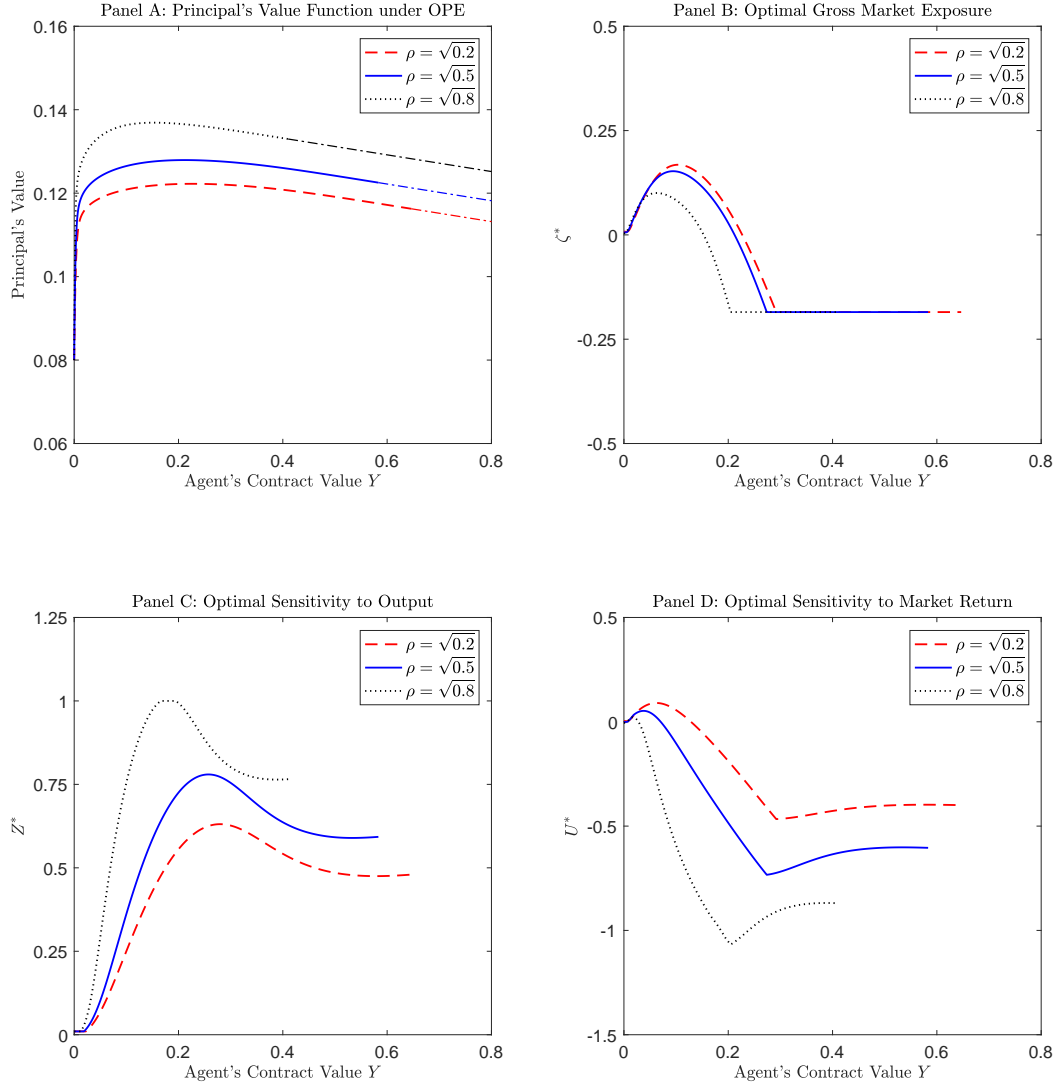
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**Table 1**  
**Summary of Key Parameters and Parameter Calibration**

Variable	Symbol	Benchmark Value
Risk-Free Rate	$r$	0.035
Expected Return on Market Portfolio	$m$	0.1
Volatility of Market Portfolio	$\sigma$	0.1625
Expected Output of the Project without Effort	$\mu$	0.1
Volatility of the Project	$\psi$	0.1625
Easiness of Hedging Parameter	$\rho$	$\sqrt{0.5}$
Agent's Risk Aversion Parameter	$\gamma$	5
Agent's Utility Discount Rate	$\bar{\delta}$	0.035
Principal's Discount Rate	$\delta$	0.02
Liquidation Value of the Project	$\Phi$	4
Effort Cost Parameter 1	$\kappa$	13
Effort Cost Parameter 2	$b$	0.01
Lower Bound of $\zeta$	$\underline{\Gamma}$	-0.185
Lower Bound of Effort	$\underline{A}$	0
Upper Bound of Effort	$\bar{A}$	0.0762

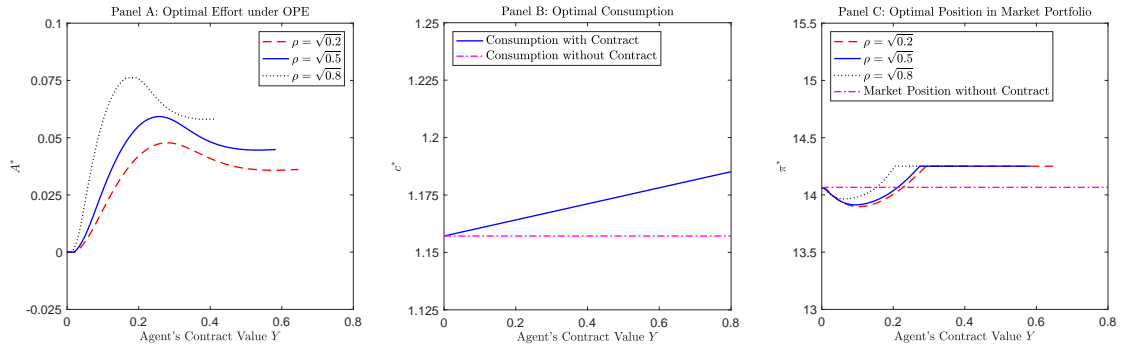
We mainly refer to He (2011) to determine the parameter values and comply with technical conditions in Section 6.6 in the Appendix. Sensitivity  $Z$  is restricted within  $[\underline{Z}, \bar{Z}]$  where  $\underline{Z} = b = 0.01$  and  $\bar{Z} = 1$ . The project's liquidation value is set at  $Q = 0.8\mu/\delta$ . We assume that face value of debt  $F$  is sufficiently large so that the residual value for stocks is 0 should the project be liquidated. We do not specify  $\bar{\Gamma}$ , the upper bound of  $\zeta$ , because it never explodes upward.

**Figure 2**  
**The OPE Contract**



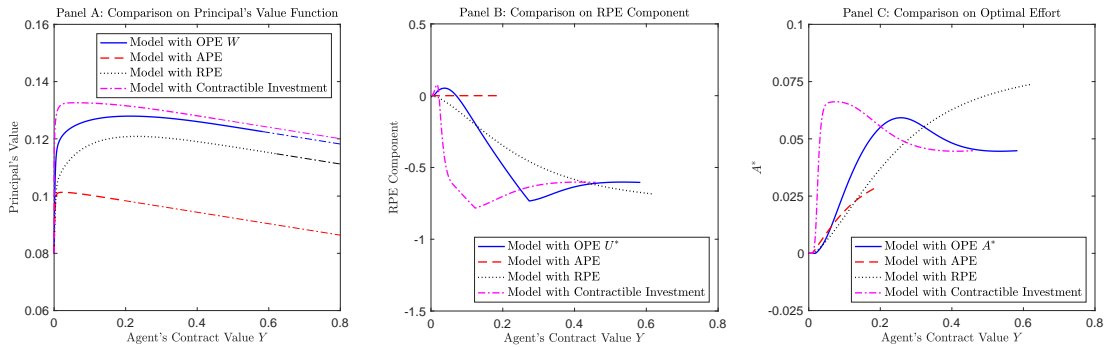
This figure plots numerical solution to the OPE contract. Panel A depicts the principal's value function  $W$  under three different values of  $\rho$ . In each case, the value function is extended by a dash-dot line with slope  $-\delta$ . Panel B and C plot optimal gross market position  $\zeta^*$  and sensitivity to output  $Z^*$ , respectively. Optimal sensitivity to the market return  $U^* = \zeta^* - \frac{\rho\psi}{\sigma}Z^*$  is plotted in Panel D. The payment boundary is  $\bar{y} = 0.6466$  when  $\rho = \sqrt{0.2}$ ,  $\bar{y} = 0.5833$  when  $\rho = \sqrt{0.5}$ , and  $\bar{y} = 0.4103$  when  $\rho = \sqrt{0.8}$ . All other parameters are the same as in Table 1.

**Figure 3**  
**The Agent's Optimal Effort, Consumption, and Portfolio Choice**



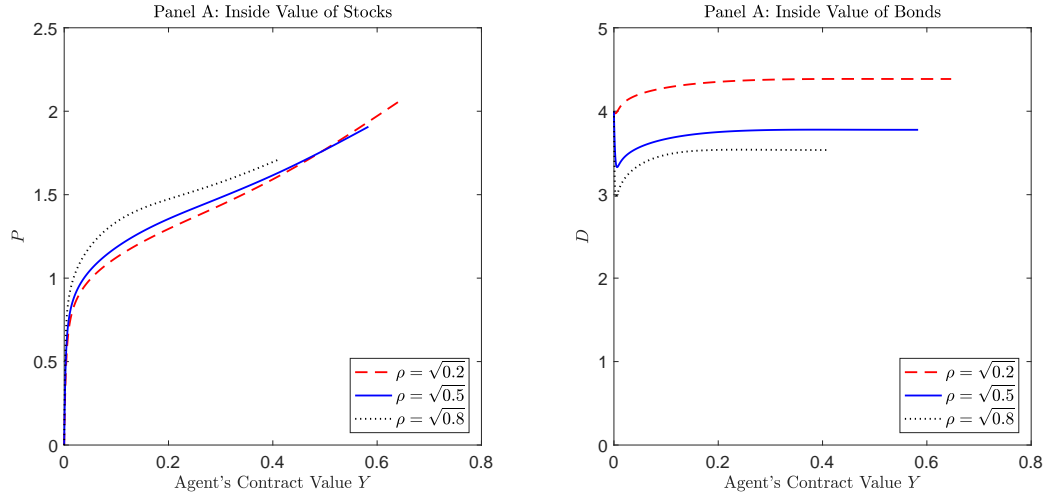
Panel A in this figure plots the agent's optimal effort choice  $A^*$  under the OPE contract. Panel B plots the agent's optimal consumption  $c^*$  with and without the contract. We set  $S \equiv 20$  so that the agent's optimal consumption rate after she retires is 1.1571. Panel C plots the agent's optimal position (in monetary unit) invested in the market portfolio  $\pi^*$ . After retirement or without the contract, the agent invests 14.0659 (in monetary unit) into the market portfolio. When the agent enters into the contract, her portfolio position has a hedging component  $-\zeta^*$ . All other parameters are the same as in Table 1.

**Figure 4**  
**Model Comparison**



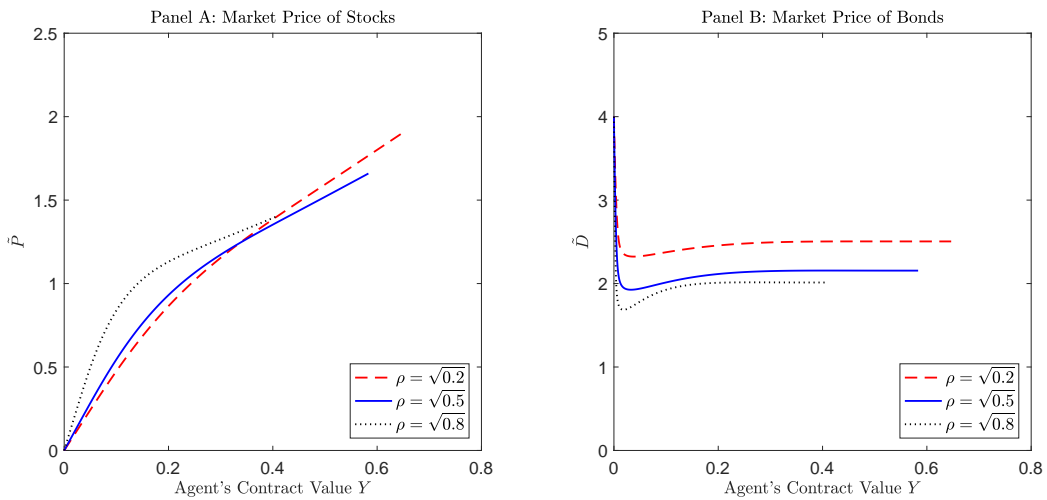
In this figure, we compare four models with the parameters in Table 1: the model with OPE, APE, RPE, and one with contractible managerial investment. In the last model, the principal imposes  $\pi = 0$  when  $W'(y) > 0$  and  $\pi = \pi^*$  when  $W'(y) \leq 0$  in the contract. Panel A compares the principal's value function. Panel B shows the RPE component across models. Panel C demonstrates the agent's optimal effort choice. The payment boundary is  $\bar{y} = 0.5833$  in the OPE contract,  $\dot{y} = 0.1955$  in the APE contract,  $\ddot{y} = 0.6218$  in the RPE contract, and  $\tilde{y} = 0.4589$  in the contract with contractible managerial investment.

**Figure 5**  
**Inside Values of Securities**



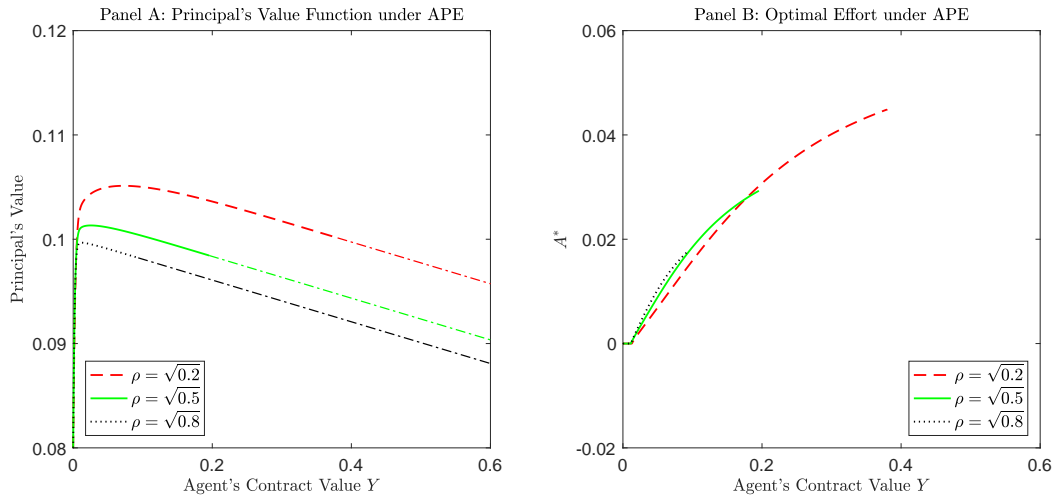
This figure plots inside value of stocks  $P$  in Panel A and that of bonds  $D$  in Panel B in our model implementation under three different values of  $\rho$ . All other parameters are the same as in Table 1.

**Figure 6**  
**Market Prices of Securities**



This figure plots market price of stocks  $\tilde{P}$  in Panel A and that of bonds  $\tilde{D}$  in Panel B in our model implementation under three different values of  $\rho$ . All other parameters are the same as in Table 1.

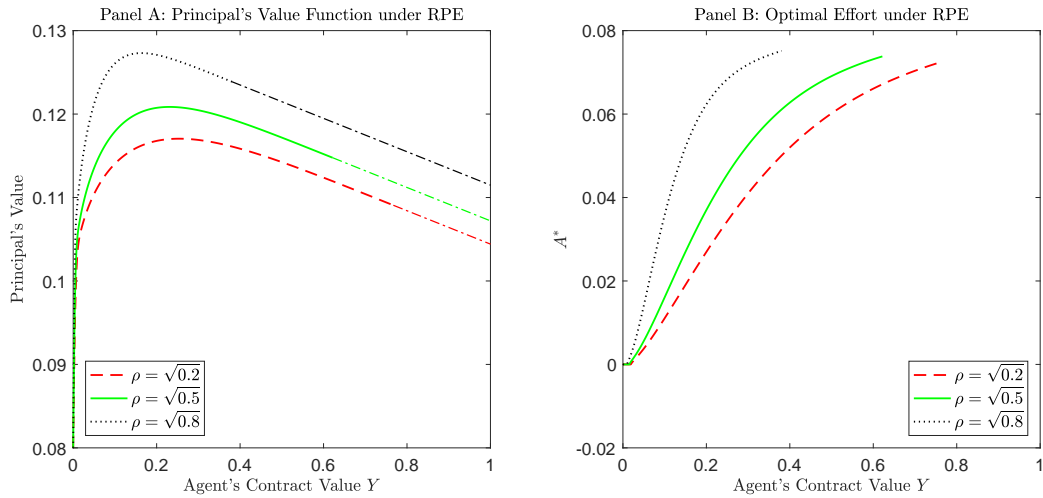
**Figure A1**  
**The APE Contract**



This figure plots numerical solution to the model with APE under three different values of  $\rho$ . Panel A shows the principal's value function. In each case, the value function is extended by a dash-dot line with slope  $-\delta$ . Panel B shows correspondingly the agent's optimal effort. The payment boundary is  $\hat{y} = 0.3802$  when  $\rho = \sqrt{0.2}$ ,  $\hat{y} = 0.1955$  when  $\rho = \sqrt{0.5}$ , and  $\hat{y} = 0.0935$  when  $\rho = \sqrt{0.8}$ . All other parameters are the same as in Table 1.



**Figure A2**  
**The RPE Contract**



This figure plots numerical solution to the model with RPE under three different values of  $\rho$ . Panel A shows the principal's value function. In each case, the value function is extended by a dash-dot line with slope  $-\delta$ . Panel B shows correspondingly the agent's optimal effort. The payment boundary is  $\bar{y} = 0.7673$  when  $\rho = \sqrt{0.2}$ ,  $\bar{y} = 0.6218$  when  $\rho = \sqrt{0.5}$ , and  $\bar{y} = 0.3807$  when  $\rho = \sqrt{0.8}$ . All other parameters are the same as in Table 1.